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Peter O'Brien and John L. Wyatt, Jr.

Abstract

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RC tree networks. Both estimates and bounds are derived for the step response
of "leaky" RC trees and meshes, a class of networks that is appropriate for
modelling interconnect in digital bipolar circuits. This paper is intended to
serve as a tutorial as well as a research report. Therefore existing work is
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Signal Delay in Leaky RC Mesh Models for Bipolar Interconnect

Peter O'Brien and John L. Wyatt, Jr.[**]

This paper extends the results of Penfield and Rubinstein on signal delay in RC tree networks. Both estimates and bounds are derived for the step response of "leaky" RC trees and meshes, a class of networks that is appropriate for modelling interconnect in digital bipolar circuits. This paper is intended to serve as a tutorial as well as a research report. Therefore existing work is explained in some detail along with the derivation of new results.

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Contents

I. Introduction

Simple closed-form bounds for signal propagation delay in linear

RC tree models for interconnect were derived in [1]. The theory has

since been extended to include RC mesh models as well [2]. The goal of

this report is to further extend the theory to allow for RC networks with

grounded resistors (i.e., "leaky" RC trees and meshes -- see section II

for a precise definition of these terms). Both delay estimates and delay

bounds are developed.

The practical motivation behind allowing resistive paths to ground

in RC mesh circuits is an attempt to more accurately model interconnect

on bipolar chips. The base of a bipolar transistor loading the interconnect

offers a (nonlinear) resistive path to ground. A corresponding gate

electrode of an MOS transistor would not offer such a resistive path to

ground, and the original work of Penfield et. al. relied upon the absence

of such paths.
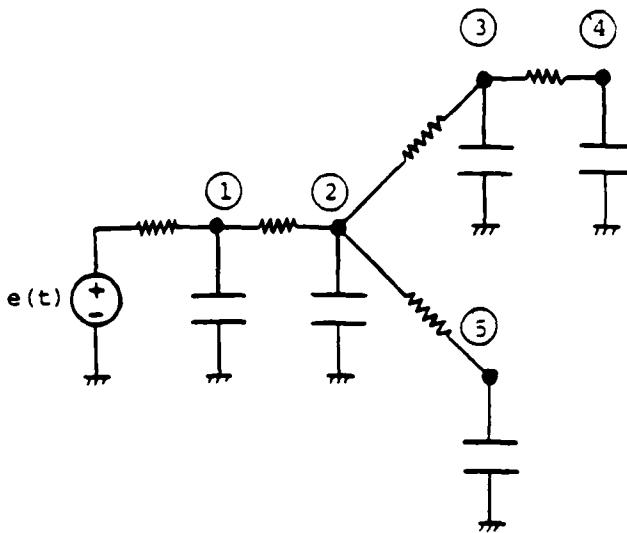
Fig. 1: Lumped linear RC tree model for MOS logic with fanout
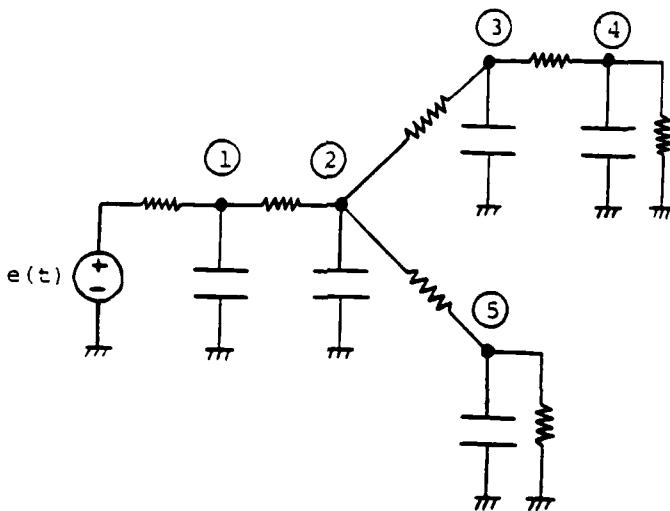
Fig. 2: Leaky RC tree model for bipolar logic with fanout

Consider the RC tree in Fig. 1. The voltage source e(t) represents
the output of some logic gate. The output of this gate fans out through
the interconnect to the inputs of two other logic gates at nodes ④ and
⑤ . Since nodes ④ and ⑤ represent the gate terminals of MOS
transistors, there is no need for a resistive path to ground in the model.

The analogous situation in bipolar logic is illustrated in Fig. 2.
Here nodes ④ and ⑤ represent the base terminals of bipolar transistors,
so we need a resistive path to ground in the model. The resulting circuit
is not an RC mesh, but would be one except for the resistive paths to ground.
This is the type of network discussed in this report. The organization
of this report can be seen in the table of contents.

II. Classes of Networks Considered

A lumped linear RC mesh is any network consisting of 2-terminal
linear resistors and capacitors driven by independent sources, such
that one side of every capacitor and every source is grounded, no
resistor is grounded, and between any two nodes other than ground there
exists a path consisting entirely of resistors. An RC mesh is called
an RC tree if it does not contain a closed path of resistors.

However, in this paper we shall consider more general types of
circuits: leaky RC meshes and trees. A leaky RC mesh is any network
meeting all of the requirements of an RC mesh given above, except for
the restriction that there be no grounded resistors. So for every leaky
RC mesh, there is an underlying RC mesh which is obtained by open-
circuiting all of the grounded resistors in the original leaky RC mesh.
A network is a leaky RC tree if it is a leaky RC mesh whose underlying
RC mesh is an RC tree (i.e., there are no resistor loops in the underlying
RC mesh).

III. Network Matrices

In this section, we define certain network matrices to be used
later in the report. Well-known facts about these matrices will be
presented. We shall motivate these definitions by the example circuit
in Fig. 3. Note that this example circuit would be an RC mesh without
the grounded resistor R_6.
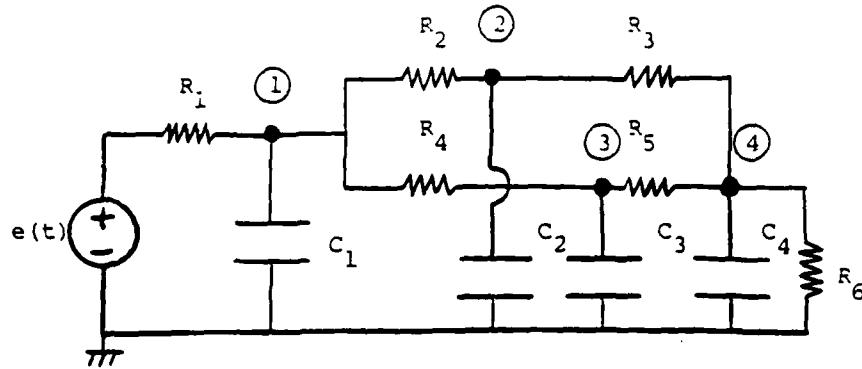
Fig. 3: Lumped linear RC network

The ground node and the node connected to the independent source are not numbered. The remaining nodes are numbered in any order from ① to Ⓝ , where N is the total number of capacitors (in Fig. 3, N=4).

a.) The resistance matrix R:
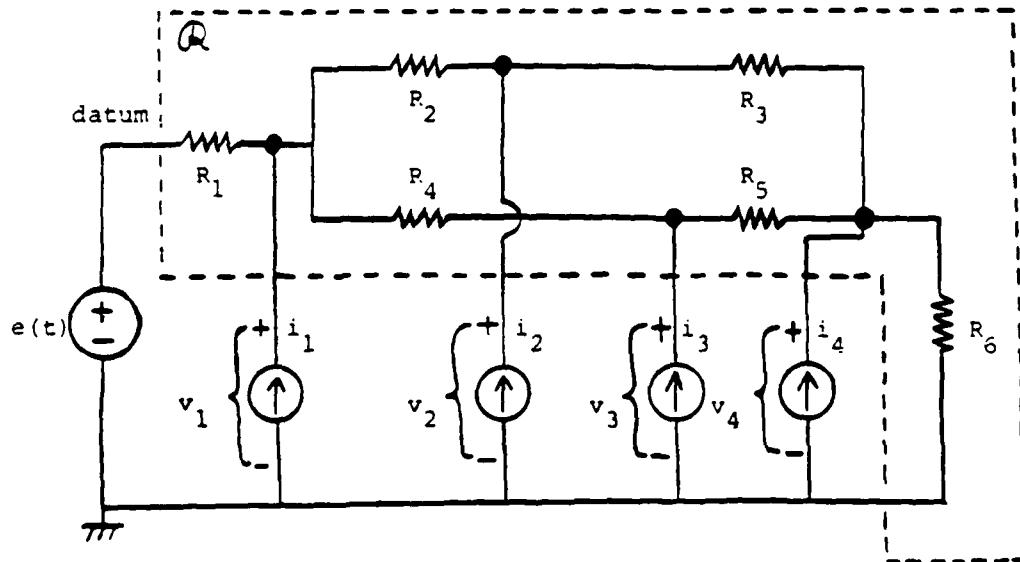
Fig. 4: The resistor subnetwork R extracted from the circuit in Fig. 3 (driven with current sources)

We isolate the resistor subnetwork R containing all the resistors and assign reference directions to the capacitor currents $i_1, \ldots, i_N$ as shown in Fig. 4 (note that positive current is defined to be flowing out of the capacitor and into the resistor subnetwork). Let the node connected to the independent source serve as the datum node of R. Define the matrix R as follows:

With the datum grounded, the node voltages with respect to ground are given in terms of the capacitor currents by the <u>resistance</u> <u>matrix</u> R as shown below:

$$
\begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{pmatrix}
=
\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{bmatrix}
\begin{pmatrix} i_1(t) \\ i_2(t) \\ \vdots \\ i_N(t) \end{pmatrix}
\tag{1}
$$

We can introduce vector notation to write (1) in a more compact form. Let $\underline{v}$ denote the column vector of node voltages $(v_1, v_2, \ldots, v_N)^T$. Let $\underline{i}$ denote the column vector of capacitor currents $(i_1, i_2, \ldots, i_N)^T$. Then (1) becomes:

$$
\underline{v}(t) = \underline{R}\, \underline{i}(t)
\tag{2}
$$

So with the datum grounded, the numerical value of any matrix entry R_{ij}, in ohms, is equal to the potential in volts we would observe at node ⓘ if a 1-amp current were injected into node ⓙ and all external nodes of R other than ⓙ and the datum were open-circuited.

In the special case of an RC tree the matrix entries can be read off by inspection, since in that case R_{ij} is simply the sum of the resistances along the route obtained by intersecting the path from node ⓘ to datum with the path from node ⓙ to datum. So, the topological definition of R_{ij} for an RC tree given in [1] is consistent with the more general interpretation of R_{ij} as an element of the resistance matrix $\underline{R}$ describing the resistor subnetwork R.

From circuit theory, the resistance matrix $\underline{R}$ has certain well-known properties independent of the particular resistor subnetwork being described by $\underline{R}$:

(i) $\underline{R}$ is symmetric (i.e., $R_{ij} = R_{ji}$ for all i,j): This is a consequence of R being a "reciprocal" network. [3; Chapter 16]

(ii) $\underline{R}$ is positive-definite (assuming all the resistors in R are positive): This is true because the power dissipated in R, given by the quadratic form $\underline{i}^T \underline{R} \, \underline{i}$ is positive for any nonzero choice of currents $\underline{i}$.

These properties of $\underline{R}$ are quite standard and play only a minor role in the theory. But $\underline{R}$ possesses more structure than symmetry and positive-definiteness, and this additional structure, described in Section VI and proved in Appendix A, is the foundation for the theory developed in this report.

b.) The conductance matrix G:
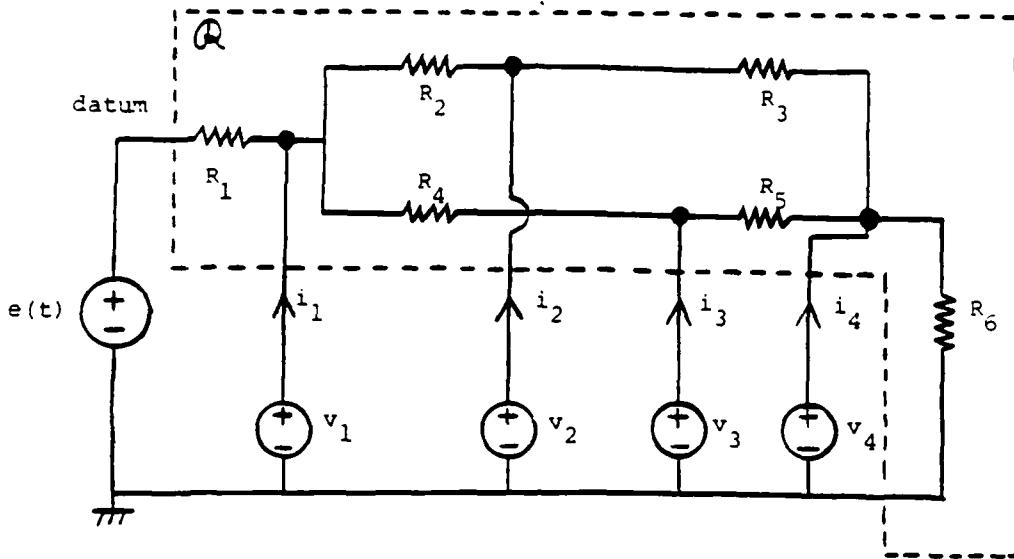
Fig. 5: The resistor subnetwork R extracted from the circuit
in Fig. 3 (driven with voltage sources)

Again, we isolate the resistor subnetwork R (Fig. 5). With the
datum grounded, we now express the capacitor currents in terms of the
node voltages by the conductance matrix G as shown below:

$$
\begin{bmatrix} i_1(t) \\ i_2(t) \\ \vdots \\ i_N(t) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{iN} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{bmatrix}
\tag{3}
$$

or, using the vector shorthand:

$$
i(t) = G \, v(t) \tag{4}
$$

So with the datum grounded, the numerical value of any matrix entry G_{ij},
in mhos, is equal to the current in amps we would observe flowing into

node (i) if a 1-volt source were applied to node (j) and all external nodes of R other than (j) were grounded.

If R is the resistance matrix for the same resistor subnetwork described by G, then $G = R^{-1}$. Like R, G is also positive-definite symmetric (for similar reasons as R), and G possesses additional structure described in Appendix A which is crucial to the results in Section VI.

c.) The capacitance matrix C:

This matrix is not as central to the theory as either the R or the G matrix. It is simply notational shorthand for writing down the capacitors in the RC network under consideration. We define $C \overset{\Delta}{=} \text{diag}\{C_1, C_2, \ldots, C_N\}$. So C is a "diagonal" matrix, (i.e., has off-diagonal elements equal to zero), and the diagonal elements are equal to the capacitor values consistent with the assigned node numbers.

One simple property to note about C is that its inverse, C^{-1}, is also diagonal: $C^{-1} \overset{\Delta}{=} \text{diag}\{C_1^{-1}, C_2^{-1}, \ldots, C_N^{-1}\}$. (Of course, C^{-1} doesn't exist in the degenerate case where some individual capacitors are equal to zero.)

IV. Network Differential Equations in Canonical Form

In order to write down the network differential equations, we must first consider what happens when the datum, e(t), is not grounded. In the previous section we wrote the node voltages in terms of the capacitor currents when the datum was grounded:

$$v(t) = R\, i(t) \; ; \; (\text{datum } e(t) = 0) \tag{5}.$$

When $e(t)$ is not zero, then by superposition we can write:

$$v(t) = R\, i(t) + x e(t) \tag{6},$$

where $x = (x_1, x_2, \ldots, x_N)^T$ is a column vector of dimensionless numbers (each x_i satisfying the inequality: $0 < x_i \leq 1$). Referring back to Fig. 4, each x_i is numerically equal to the potential (in volts) produced at node ⓘ due solely to a 1-volt source at the datum, while open-circuiting all of the external current sources.

Example: In Fig. 3, if $R_6 = \infty$ (open-circuit), the circuit is an RC mesh and $x = (1\ 1\ 1\ 1)^T$.

Example: In Fig. 3, if R_6 is some finite resistance, x requires some computation. The result is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{R_1 + R_6 + (R_2 + R_3) \;\|\; (R_4 + R_5)} \begin{bmatrix} R_6 + (R_2 + R_3) \;\|\; (R_4 + R_5) \\[2ex] R_6 + \dfrac{R_3(R_4 + R_5)}{R_2 + R_3 + R_4 + R_5} \\[2ex] R_6 + \dfrac{R_5(R_2 + R_3)}{R_2 + R_3 + R_4 + R_5} \\[2ex] R_6 \end{bmatrix}$$

The general computational problem of calculating x is discussed in Section VIII.

To begin the formulation of the network differential equations, we start by rewriting equation (6),

$$x \; e(t) \; - \; v(t) \; = \; -R \; i(t) \tag{7}.$$

The constitutive relations for the capacitors can be neatly written as:

$$i(t) \; = \; -C \; \dot{v}(t) \tag{8},$$

where the minus sign arises from our sign convention for capacitor currents. Substituting (3) into (7) we have,

$$\boxed{x \; e(t) \; - \; v(t) \; = \; [R \; C] \; \dot{v}(t)} \tag{9}.$$

The system of equations in (9) is the canonical form for leaky RC meshes. The equations in (9) are also the most general: valid for all time and for any possible e(t). At this point, it is worthwhile specializing the system in (9) somewhat by considering only a certain form for e(t).

Since the voltage e(t) models the output of a logic gate undergoing a logic transition, typically e(t) reaches and remains at its final value, $e(\infty)$, for some time interval t≥T.

Example:
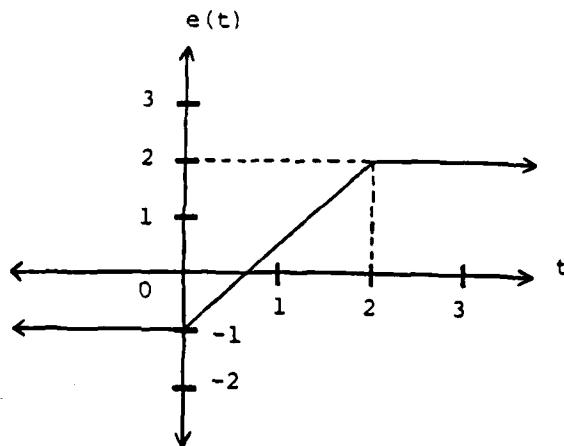
$$\text{final value} = e(\infty) = 2$$
$$T = 2$$

Example: $e(t)$ = "unit step at the origin"
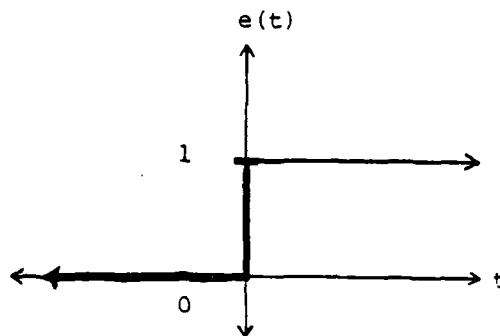
$$\text{final value} = e(\infty) = 1$$
$$T = 0$$

For an $e(t)$ of this form, we can replace $e(t)$ by $e(\infty)$ for $t \geq T$. So (9) becomes:

$$\underset{\sim}{x} e(\infty) - \underset{\sim}{v}(t) = [RC] \underset{\sim}{\dot{v}}(t) , \qquad (10)$$

valid for a "saturating" $e(t)$ for $t \geq T$.

Now we can identify $\underset{\sim}{x} e(\infty)$ as the final equilibrium voltage distribution.

At equilibrium $\dot{v}(t) = 0$, so (10) becomes:

$$\underset{\sim}{x}e(\infty) - \underset{\sim eq}{v} = [RC]\underset{\sim\sim}{0} = \underset{\sim}{0} \Rightarrow \boxed{\underset{\sim eq}{v} = \underset{\sim}{x}e(\infty)} \qquad (11)$$

Finally, since estimates for and bounds on the step response will be derived later in this report, the canonical equations for a step input will be given. Using (10) and (11),

$$\boxed{\begin{array}{l} \text{Network Differential Equations for a Step Input:} \\[2em] \underset{\sim eq}{v} - \underset{\sim}{v}(t) = [RC]\underset{\sim\sim}{\dot{v}}(t) \; ; \; t \geq 0 \end{array}} \qquad (12)$$

V. Delay Estimation

In this section, a good first-order estimate of the delay through a leaky RC mesh is developed. This delay estimate is called the "Elmore Delay" in the literature [4], also [1], [5-7]. We will denote it here by "T_{D_i}". By convention, we call the output node of interest "node $\textcircled{i}$." So the subscript on T_{D_i} simply indicates that the delay estimate depends on which node in the RC network we have chosen as our output node. Finally, since T_{D_i} is just a single number, the use of T_{D_i} alone takes no account of the threshold voltage of the device driven by node $\textcircled{i}$. A simple procedure for accounting for the logic threshold is presented.

a.) Definition of T_{D_i}:

We define T_{D_i} in terms of the response at the output node $\textcircled{i}$ to a step transition at the input. We allow for a non-zero initial

equilibrium at the time of the step input, and denote the <u>initial equilibrium voltage distribution</u> with the vector $v_{-o} = (v_{1_o}, v_{2_o}, \ldots, v_{N_o})^T$. Using the same notation as the previous section, we denote the <u>final equilibrium voltage distribution</u> with the vector $v_{-eq} = (v_{1_{eq}}, v_{2_{eq}}, \ldots, v_{N_{eq}})^T$. The step transition can be either a step increase $(v_{-eq} > v_{-o})$ or a step decrease $(v_{-eq} < v_{-o})$. The amplitude of the step <u>input</u> is equal to $\frac{1}{x_i}(v_{i_{eq}} - v_{i_o})$.

<u>Definition</u>:

$$T_{D_i} \overset{\Delta}{=} \int_0^\infty \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \, dt \, , \qquad (13)$$

where $v_i(t)$ is the response at node $\textcircled{i}$ to a step transition at the input at time t=0 of amplitude $\frac{1}{x_i}(v_{i_{eq}} - v_{i_o})$.

Graphically, T_{D_i} is shown below:
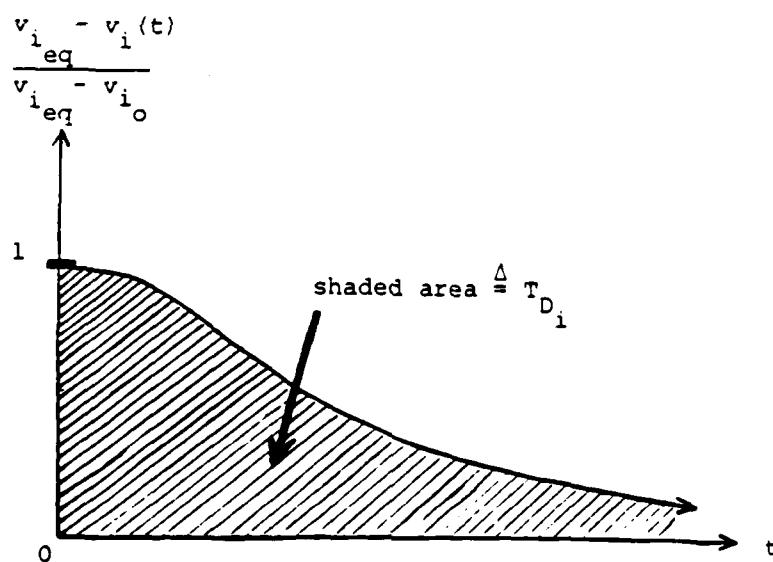
Fig. 6: Definition of T_{D_i}

b.) <u>Computation of T_{D_i}</u>:

Rewriting the canonical matrix equation for a step input, we have:

$$\underset{\sim}{v}_{eq} - \underset{\sim}{v}(t) = [RC]\underset{\sim\sim}{\dot{v}}(t) \; ; \; t \geq 0 \tag{14}.$$

Taking the ith row of this matrix equation,

$$v_{i_{eq}} - v_i(t) = \sum_k R_{ik}C_k\dot{v}_k(t) \; ; \; t \geq 0 \tag{15}$$

(where $\sum_k$ is taken over all capacitor nodes in the RC network).

Dividing by $v_{i_{eq}} - v_{i_o}$ and then integrating from t=0 to t=∞, we obtain T_{D_i}:

$$T_{D_i} \overset{\Delta}{=} \int_0^\infty \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \; dt = \frac{\sum_k R_{ik}C_k \int_0^\infty \dot{v}_k(t) \; dt}{v_{i_{eq}} - v_{i_o}}$$

$$= \boxed{\frac{\sum_k R_{ik}C_k(v_{k_{eq}} - v_{k_o})}{v_{i_{eq}} - v_{i_o}}} \tag{16}.$$

So in order to compute T_{D_i} we need to know:

1) The initial and final equilibrium voltage distributions, $\underset{\sim}{v}_o$ and $\underset{\sim}{v}_{eq}$. Note that this is really equivalent to knowing the <u>single</u> vector $\underset{\sim}{x}$, since for a step input we have $\underset{\sim}{v}_o = \underset{\sim}{x}e(0^-)$ and $\underset{\sim}{v}_{eq} = \underset{\sim}{x}e(0^+)$. In fact, we can express T_{D_i} in terms of $\underset{\sim}{x}$ instead of $\underset{\sim}{v}_o$ and $\underset{\sim}{v}_{eq}$:

$$\boxed{T_{D_i} = \frac{1}{x_i} \sum_k R_{ik}C_kx_k} \tag{17}.$$

2) The ith row (or column) of the resistance matrix $\underset{\sim}{R}$.

3) The capacitance matrix $\underset{\sim}{C}$ (this requires no computation).

Note that when we are dealing with an RC mesh (i.e., no resistive paths to ground) starting from a zero initial equilibrium, then $\underset{\sim}{v}_o = \underset{\sim}{0}$ and $v_{i_{eq}} = v_{k_{eq}}$ for all $k \neq i$. Hence T_{D_i} reduces to $\sum_k R_{ik} C_k$, which is consistent with the definition given in earlier works [1], [2].

c.) <u>Alternate Interpretation of T_{D_i}</u> :

We have already defined T_{D_i} in terms of the response observed at node (i) , $v_i(t)$, to a step input applied at time t=0, when the RC network is at some (possibly non-zero) equilibrium. Alternately, we can interpret T_{D_i} in terms of $\dot{v}_i(t)$. Note that $\dot{v}_i(t)$, in addition to being the time derivative of $v_i(t)$ is also the response observed at node (i) to an <u>impulse</u> input applied at time t=0, when the RC network is at <u>zero equilibrium</u>.

We will show in the next section that the step response $v_i(t)$ is monotone, which means that $\dot{v}_i(t)$ can be interpreted as a density function for a continuous random variable, when properly normalized:

$$(\dot{v}_i(t))_{normalized} = \left(\frac{\dot{v}_i(t)}{\int_0^\infty \dot{v}_i(t)\,dt} \right) = \left(\frac{\dot{v}_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \tag{18}.$$

T_{D_i} is the <u>first moment</u> (or "<u>center of mass</u>") of this normalized zero-state impulse response, as shown below:

Integrating by parts,

$$\int_0^\infty t \left(\frac{\dot{v}_i(t)}{v_{i_{eq}} - v_{i_o}} \right) dt = \int_0^\infty \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}}\,dt - \left[t \left(\frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \right] \Bigg|_{t=0}^{t=\infty}$$

$$= T_{D_i} \qquad \qquad \underbrace{}_{0} \tag{19},$$

where we have used the fact that $\lim\limits_{t\to\infty} t(v_{i_{eq}} - v_i(t))=0$ because $v_i(t) \to v_{i_{eq}}$ exponentially as $t\to\infty$.

d.) Incorporating Logic Threshold:

As mentioned earlier, the use of T_{D_i} alone takes no account of the threshold voltage of the device driven by node ⓘ . To do so we need an estimate for the entire step response voltage waveform $v_i(t)$. A reasonable estimate is

$$\hat{v}_i(t) \overset{\Delta}{=} v_{i_{eq}} + (v_{i_o} - v_{i_{eq}})e^{-t/T_{D_i}} \tag{20}$$

since, like the exact but unknown response:

1) $\hat{v}_i(t)$ starts at v_{i_o} .

2) $\hat{v}_i(t)$ is monotone.

3) $\hat{v}_i(t)$ asymptotes to $v_{i_{eq}}$.

4) $\displaystyle\int_0^\infty \frac{v_{i_{eq}} - \hat{v}_i(t)}{v_{i_{eq}} - v_{i_o}}\, dt = T_{D_i}$.

In the next section we will show that $\hat{v}_i(t)$ always lies between the best currently available upper and lower bounds on the exact step response $v_i(t)$.

We can invert this estimate of the entire waveform in order to estimate the time to reach a given logic threshold voltage v_T:

$$v_T = v_{i_{eq}} + (v_{i_o} - v_{i_{eq}})e^{-t/T_{D_i}} \implies \boxed{\, t = T_{D_i}\, \ln\left(\frac{v_{i_{eq}} - v_{i_o}}{v_{i_{eq}} - v_T}\right)\,} \tag{21}.$$

Example: If the logic threshold of the device driven by node ⓘ is

$v_T = v_{i_o} + \frac{1}{2}\, (v_{i_{eq}} - v_{i_o})$, then instead of using T_{D_i} as a delay estimate,

we use the somewhat lower estimate, $T_{D_i} \ln 2 \approx 0.69 \, T_{D_i}$.

<u>Example</u>: If the logic threshold of the device driven by node (i) is

$v_T = v_{i_o} + (1 - \frac{1}{e}) \, (v_{i_{eq}} - v_{i_o}) \approx v_{i_o} + 0.63 \, (v_{i_{eq}} - v_{i_o})$, then we use

exactly T_{D_i} as a delay estimate.

The estimate (20) is not generally exact: for one thing, it contains only a single natural frequency: $-\frac{1}{T_{D_i}}$. To better appreciate the nature of the approximation involved, Horowitz [7] has noted that, in the case of <u>no grounded resistors</u>, a sufficient condition for (20) to be exact is that the time derivatives of all the capacitor voltages are equal: i.e., $\dot{v}_k(t) = \dot{v}_i(t)$ for all k. In the case where we <u>do have grounded resistors</u>, the estimate (20) is exact when the <u>normalized</u> time derivatives of all the capacitor voltages are equal:

$$\frac{\dot{v}_k(t)}{v_{k_{eq}} - v_{k_o}} = \frac{\dot{v}_i(t)}{v_{i_{eq}} - v_{i_o}} \quad \text{for all } k \neq i. \tag{22}$$

We can see this by going back to the ith row of the original system of equations for the node voltages:

$$v_{i_{eq}} - v_i(t) = \sum_k R_{ik} C_k \dot{v}_k(t)$$

$$= \sum_k R_{ik} C_k \, \frac{v_{k_{eq}} - v_{k_o}}{v_{i_{eq}} - v_{i_o}} \, \dot{v}_i(t) \quad ; \text{ using the approximation}$$
$$\text{in (22)}$$

$$= \left[\frac{\sum_k R_{ik} C_k (v_{k_{eq}} - v_{k_o})}{v_{i_{eq}} - v_{i_o}} \right] \dot{v}_i(t)$$

$$= T_{D_i} \dot{v}_i(t) \tag{23}$$

The solution to the first-order differential equation given by (23) is

$$v_i(t) = \hat{v}_i(t).$$

Another view of the nature of the approximation (20) will emerge from consideration of the step response bounds, which we consider next.

VI.   Optimal Control Method for Determing Bounds on the Step Response

The purpose of this section is to clearly derive step response bounds for leaky RC meshes using an optimal control point-of-view. The bounds derived will be analogous to those in [1]. Recent bound-tightening results such as slew-rate limitations [8], and (in the case of RC trees) spatial convexity [9] will not be considered here. These recent improvements are omitted because they do not add to the basic understanding of the optimal control approach. We do not yet know exactly to what extent these bound-tightening results can be applied to leaky RC meshes. It is clear, however, that at least the simplest of the slew-rate limitations applies and always gives somewhat tighter bounds than those developed here.

a.) Preliminary Results:

Fact 1: For any three nodes (i) , (j) , (k) of a leaky RC mesh,

$$R_{ii}R_{kj} \geq R_{ki}R_{ij} \qquad (24),$$

where the resistances in (24) are elements of the resistance matrix R in (1).

The proof of Fact 1 depends on certain properties of the conductance matrix $\underline{G}$. These properties, as well as the proof of Fact 1, are given in Appendix A.

Fact 2: For any node ⓙ of a leaky RC mesh, the observed

response, $v_j(t)$, to a step transition at the

input applied at time t=0, when the network is

at some (possibly non-zero) equilibrium, is

<u>monotone</u>:

$$\dot{v}_j(t) \geq 0, \ \forall t \geq 0 \ ; \ \text{if } v_{j_{eq}} > v_{j_o}$$

$$\dot{v}_j(t) \leq 0, \ \forall t \geq 0 \ ; \ \text{if } v_{j_{eq}} < v_{j_o} \quad (25).$$

The proof of Fact 2 is also given in Appendix A.

Fact 3: For any two nodes ①, Ⓚ of a leaky RC mesh:

On an "up" transition $(\underline{v}_{eq} > \underline{v}_o)$ we have,

$$R_{ii}[v_{k_{eq}} - v_k(t)] \geq R_{ki}[v_{i_{eq}} - v_i(t)] \quad (26)$$

$$R_{ki}[v_{k_{eq}} - v_k(t)] \leq R_{kk}[v_{i_{eq}} - v_i(t)] \quad (27).$$

On a "down" step transition $(\underline{v}_{eq} < \underline{v}_o)$ the sense of

each inequality is reversed,

$$R_{ii}[v_{k_{eq}} - v_k(t)] \leq R_{ki}[v_{i_{eq}} - v_i(t)] \quad (28)$$

$$R_{ki}[v_{k_{eq}} - v_k(t)] \geq R_{kk}[v_{i_{eq}} - v_i(t)] \quad (29).$$

The proof of Fact 3, which uses both Fact 1 and Fact 2, is presented here in the body of the report.

Proof of Fact 3:

The system of differential equations for the node voltages is:

$$\underset{\sim}{v}_{eq} - \underset{\sim}{v}(t) = [RC]\underset{\sim}{\dot{v}}(t) \tag{30}.$$

Taking the kth row $\Longrightarrow v_{k_{eq}} - v_k(t) = \sum_j R_{kj} C_j \dot{v}_j(t) \tag{31}.$

Taking the ith row $\Longrightarrow v_{i_{eq}} - v_i(t) = \sum_j R_{ij} C_j \dot{v}_j(t) \tag{32}.$

Using (31) and (32), we compute:

$$R_{ii}[v_{k_{eq}} - v_k(t)] - R_{ki}[v_{i_{eq}} - v_i(t)] = \sum_j (R_{ii} R_{kj} - R_{ki} R_{ij}) C_j \dot{v}_j(t) \tag{33}.$$

By applying Facts 1 and 2, (26) and (28) now follow directly from (33). By interchanging the subscripts $i \leftrightarrow k$ and using the fact that $\underset{\sim}{R}$ is symmetric, (27) and (29) follow from (26) and (28) respectively. ∎

b.) A New State Variable:

At this point, we will introduce a new state variable, f_i, that will be part of the reduced order model in the formulation of the optimal control problem. We also will prove a lemma concerning f_i. The result of this lemma will serve as the state constraints for the optimal control problem. The presentation of the optimal control problem itself is delayed until later (part (c) of Section VI).

The new state variable is defined as follows:

$$f_i(t) \stackrel{\Delta}{=} \int_t^\infty \frac{v_{i_{eq}} - v_i(t')}{v_{i_{eq}} - v_{i_o}} \, dt' \qquad (34),$$

where (as usual): $\quad v_i(t)$ is the response at node $\textcircled{i}$ to a step input

at time t=0.

v_{i_o} is the initial equilibrium voltage at node $\textcircled{i}$.

$v_{i_{eq}}$ is the final equilibrium voltage at node $\textcircled{i}$.

This definition is illustrated below:
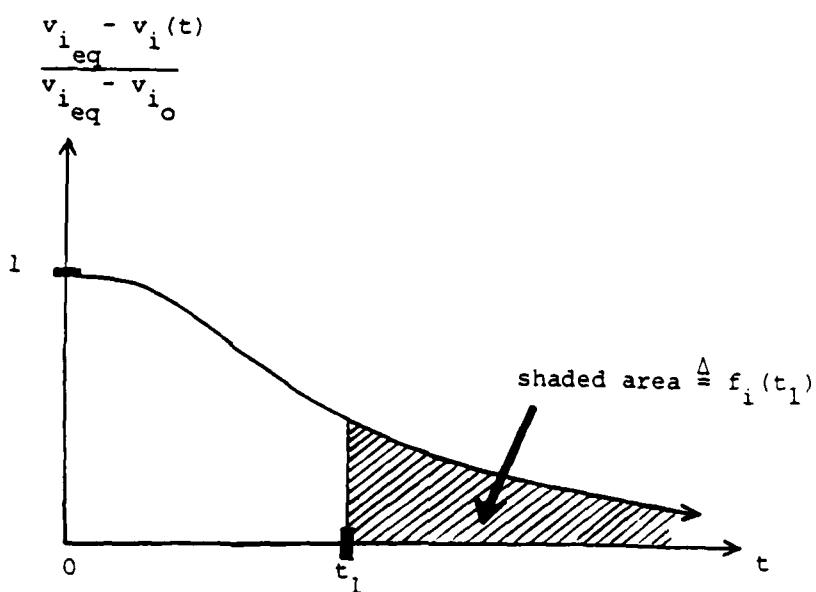
Fig. 7: Definition of f_i

Note the similarity of Fig. 7 to Fig. 6 (where we defined T_{D_i}). In fact, we can verify directly from these illustrations the initial condition:

$$f_i(0) = T_{D_i} \qquad (35).$$

Due to normalization in the denominator of (34), the behavior of $f_i(t)$ is the same for "up" steps and "down" steps of arbitrary magnitude at the input. Starting at T_{D_i} when t≈0, $f_i(t)$ will decrease monotonically towards

zero as $t \to \infty$.

> **Lemma:** For any choice of output node ⓘ in a leaky RC mesh,
>
> $$T_{R_i} \left(\frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \leq f_i(t) \leq T_p \left(\frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \qquad (36).$$
>
> where:
>
> $$T_{R_i} \triangleq \frac{1}{R_{ii}} \sum_k R_{ik}^2 C_k \qquad (37).$$
>
> $$T_p \triangleq \sum_k R_{kk} C_k \qquad (38).$$

Proof of Lemma:

Starting with the ith row of the canonical matrix differential equation,

$$v_{i_{eq}} - v_i(t') = \sum_k R_{ik} C_k \dot{v}_k(t') \qquad (39),$$

we divide by $v_{i_{eq}} - v_{i_o}$, and then integrate from $t'=t$ to $t'=\infty$ to obtain $f_i(t)$:

$$f_i(t) \triangleq \int_t^\infty \frac{v_{i_{eq}} - v_i(t')}{v_{i_{eq}} - v_{i_o}} \, dt' = \frac{\sum_k R_{ik} C_k \int_t^\infty \dot{v}_k(t') \, dt'}{v_{i_{eq}} - v_{i_o}}$$

$$= \frac{\sum_k R_{ik} C_k (v_{k_{eq}} - v_k(t))}{v_{i_{eq}} - v_{i_o}} \qquad (40).$$

Using (26) from Fact 3 on an "up" step [or (28) on a "down" step] in (40) gives:

$$f_i(t) \geq \frac{\sum_k R_{ik} C_k \left[\frac{R_{ki}}{R_{ii}} (v_{i_{eq}} - v_i(t)) \right]}{v_{i_{eq}} - v_{i_o}}$$

$$= \left(\frac{1}{R_{ii}} \sum_k R_{ik}^2 C_k \right) \left(\frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \qquad (41),$$

where we have used the fact that $R_{ki} = R_{ik}$, by symmetry of R.

Using (27) from Fact 3 on an "up" step [or (29) on a "down" step] in (40) gives:

$$f_i(t) \leq \frac{\sum\limits_k R_{ik}C_k \left[\frac{R_{kk}}{R_{ki}} (v_{i_{eq}} - v_i(t)) \right]}{v_{i_{eq}} - v_{i_o}}$$

$$= \left(\sum_k R_{kk}C_k \right) \left(\frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \tag{42},$$

where, again, we have used the fact that $R_{ki} = R_{ik}$. Together, (41) and (42) constitute (36). This proves the lemma. ∎

c.) <u>The Optimal Control Approach</u>:

The step response bounds, as in [1], can be derived by "ad hoc" manipulation of (25) and (36), but the methodology is somewhat obscure. A clearer view emerges from recasting these calculations into the form of a linear minimum - (and maximum -) time optimal control problem with state constraints. In addition, this approach has the advantage that it becomes much clearer how to incorporate additional information in order to obtain tighter bounds.

The optimal control approach is based on a reduced order model of the voltage transition process at a selected output node ⓘ . We regard this process as being described by the following <u>second</u> order model:

$$
\begin{bmatrix} \dot{f}_i(t) \\ \dot{g}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_i(t) \\ g_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{43},
$$

where:

(i) $g_i(t)$ is shorthand for the _normalized_ "_distance to go_" for the step response at node $\textcircled{i}$:

$$
\text{i.e.,} \quad g_i(t) \overset{\Delta}{=} \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \tag{44}.
$$

(ii) $f_i(t)$ is defined as in (34).

[Note that $f_i(t) \overset{\Delta}{=} \int_t^\infty \dfrac{v_{i_{eq}} - v_i(t')}{v_{i_{eq}} - v_{i_o}} \, dt' \overset{\Delta}{=} \int_t^\infty g_i(t')dt'$, so by the fundamental theorem of calculus we have $\dot{f}_i(t) = -g_i(t)$, as in (43).]

(iii) The _input_ $u(t)$ is introduced to represent the unknown waveform
$$
\dot{g}_i(t) = - \left(\frac{\dot{v}_i(t)}{v_{i_{eq}} - v_{i_o}} \right).
$$

In addition to the _dynamics_ in (43) we also have,

initial conditions (see (35)):
$$
\begin{bmatrix} f_i(0) \\ g_i(0) \end{bmatrix} = \begin{bmatrix} T_{D_i} \\ 1 \end{bmatrix} \tag{45},
$$

state constraints (see (36)):
$$
T_{R_i} g_i(t) \leq f_i(t) \leq T_P g_i(t) \tag{46},
$$

and the _input constraint_ (see (25)):
$$
u(t) \leq 0 \tag{47}.
$$

To arrive at upper and lower bounds on $g_i(t)$ we pose the following optimal control problem: What is the minimum time $t_{min}(g_i^*)$ and the maximum time $t_{max}(g_i^*)$ required for a solution of (43) to pass through a given "target" value g_i^* (with $0 < g_i^* \leq 1$) and still satisfy the constraints given by (45), (46), and (47)? Once the maximum (minimum) time $t_{max}(g_i^*)$ $(t_{min}(g_i^*))$ is known as a function of g_i^*, then the inverse function gives an upper (lower) bound on $g_i(t)$. [An upper (lower) bound on $g_i(t)$ can then be easily converted to a lower (upper) bound on the voltage step response $v_i(t)$.]

The above problem is called a "state constrained linear optimal control problem" and the Pontryagin's maximum principle could be used to find the optimal $u(t)$ and the resulting optimal state-space trajectory $(f_i(t), g_i(t))$. However, the problem can be solved in a much simpler way as follows. Regard any trajectory in the f_i-g_i plane as a function of f_i, i.e., $g_i(f_i)$. From (43) we have

$$dt = -\frac{1}{g_i} df_i \qquad (48),$$

and the time required by the trajectory to pass through two points $(f_i(0), g_i(0))$ and $(f_i(T), g_i(T))$ is

$$T = \int_0^T dt = \int_{f_i(0)}^{f_i(T)} -\frac{1}{g_i(f_i)} df_i = \left| \int_{f_i(T)}^{f_i(0)} \frac{1}{g_i(f_i)} df_i \right| \qquad (49).$$
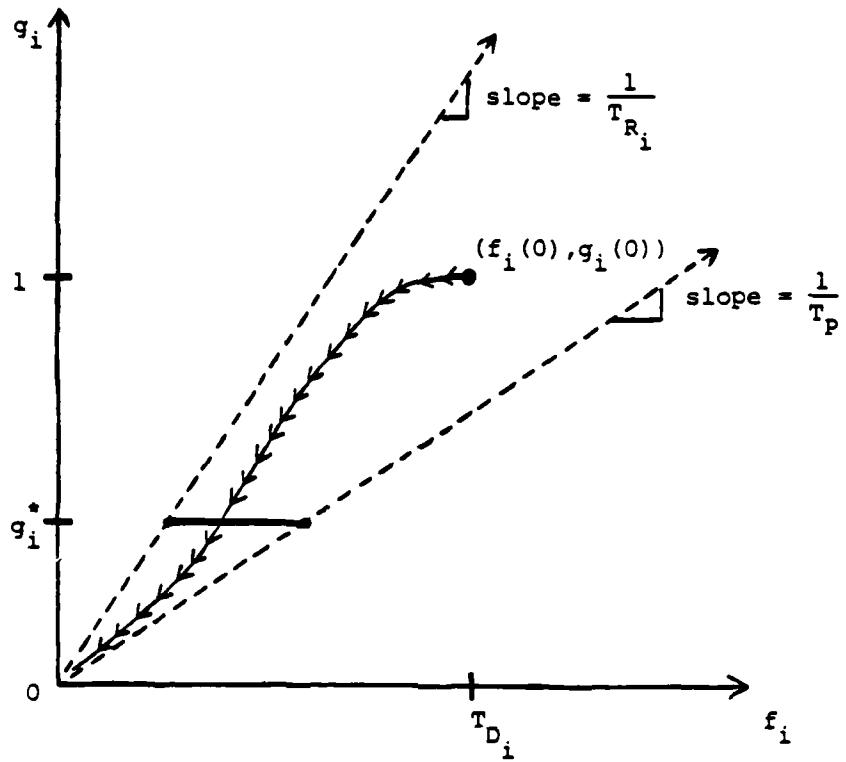
Fig. 8: "Typical" trajectory in the f_i-g_i plane

The trajectory corresponding to a voltage transition at node (i) starts at $(f_i(0), g_i(0)) = (T_{D_i}, 1)$ and because $\dot{f}_i(t) \leq 0$ and $\dot{g}_i(t) \leq 0$, it moves down and to the left towards the origin (see Fig. 8). The trajectory is confined to the region between $g_i = \frac{1}{T_p} f_i$ and $g_i = \frac{1}{T_{R_i}} f_i$ as stated by the state constraints (46). From (49), it can be seen that for $g_i(t)$ to change from 1 to a particular value g_i^*, the lowest (highest) possible trajectory (i.e., with the minimum (maximum) possible $g_i(t)$) over the widest (narrowest) integration interval on the f_i-axis requires maximum (minimum) time and therefore is the slowest (fastest) trajectory. We can

then calculate $t_{max}(g_i^*)$ and $t_{min}(g_i^*)$ for the optimal trajectories, using (49).

Solution to the maximum-time problem:

There are two different algebraic forms for $t_{max}(g_i^*)$ over the range $0 < g_i^* \leq 1$. This is a result of the fact that the maximum-time state space trajectory assumes a more complex form as g_i^* is decreased below $\dfrac{T_{D_i}}{T_P}$.

Case #1 $\left(\dfrac{T_{D_i}}{T_P} \leq g_i^* \leq 1 \right)$:
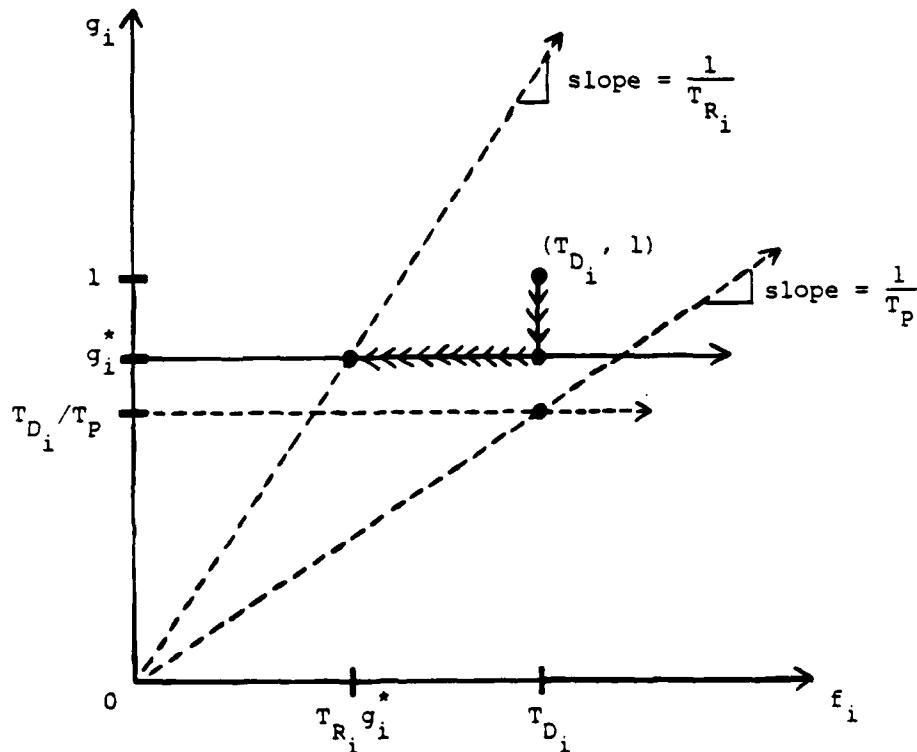
Fig. 9: Maximum-Time Trajectory (for $\dfrac{T_{D_i}}{T_P} \leq g_i^* \leq 1$)

using (49),

$$t_{max}(g_i^*) = \int_{T_{R_i}g_i^*}^{T_{D_i}} \frac{1}{g_i^*} \, df_i = \frac{1}{g_i^*} \int_{T_{R_i}g_i^*}^{T_{D_i}} df_i = \frac{1}{g_i^*} \left[T_{D_i} - T_{R_i}g_i^* \right]$$

$$= \boxed{\frac{T_{D_i}}{g_i^*} - T_{R_i}} \qquad (50).$$

<u>Case #2</u> $\quad (0 < g_i^* \leq \frac{T_{D_i}}{T_P})$:
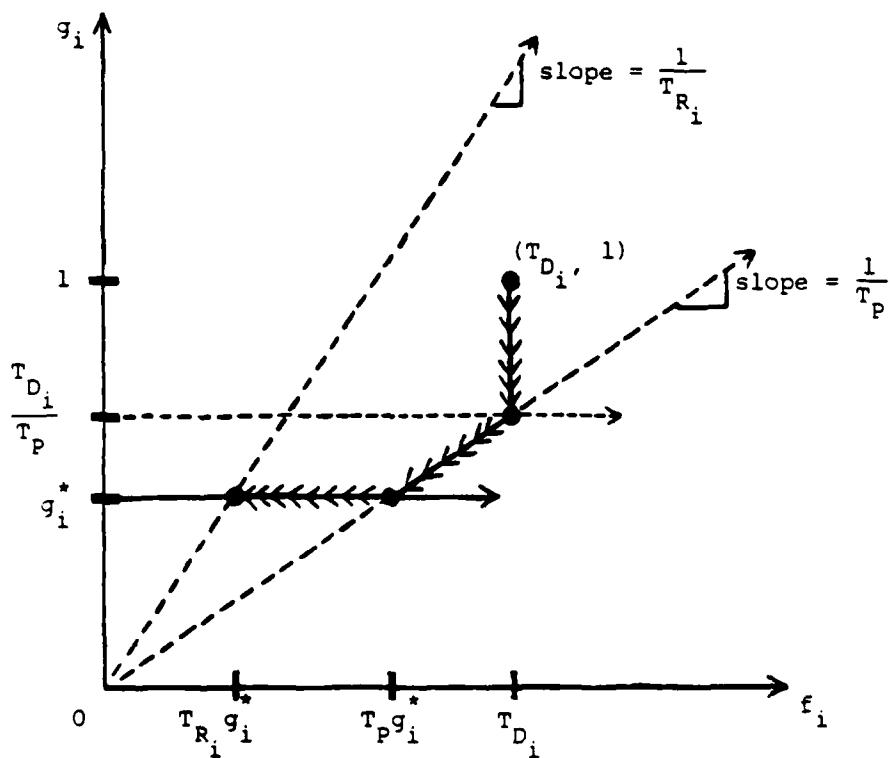
Fig. 10: Maximum-Time Trajectory (for $0 < g_i^* \leq \frac{T_{D_i}}{T_P}$)

using (49),

$$t_{max}(g_i^*) = \int_{T_{R_i}g_i^*}^{T_p g_i^*} \frac{1}{g_i^*} \, df_i + \int_{T_p g_i^*}^{T_{D_i}} -\frac{1}{\left(\frac{f_i}{T_p}\right)} \, df_i$$

$$= \frac{1}{g_i^*} \int_{T_{R_i}g_i^*}^{T_p g_i^*} df_i + T_p \int_{T_p g_i^*}^{T_{D_i}} \frac{1}{f_i} \, df_i$$

$$= \frac{1}{g_i^*} [T_p g_i^* - T_{R_i} g_i^*] + T_p [\ln(T_{D_i}) - \ln(T_p g_i^*)]$$

$$= \boxed{T_p - T_{R_i} + T_p \ln \left(\frac{T_{D_i}}{T_p g_i^*}\right)} \qquad (51).$$

Solution to the minimum-time problem:

There are three different algebraic forms for $t_{min}(g_i^*)$ over the range $0 < g_i \leq 1$. This is a result of the fact that the minimum-time state space trajectory assumes a progressively more complex form as g_i^* is decreased below $\frac{T_{D_i}}{T_p}$ and $\frac{T_{R_i}}{T_p}$.

$\underline{\text{Case \#1}} \left(\dfrac{T_{D_i}}{T_P} \leq g_i^* \leq 1 \right) :$
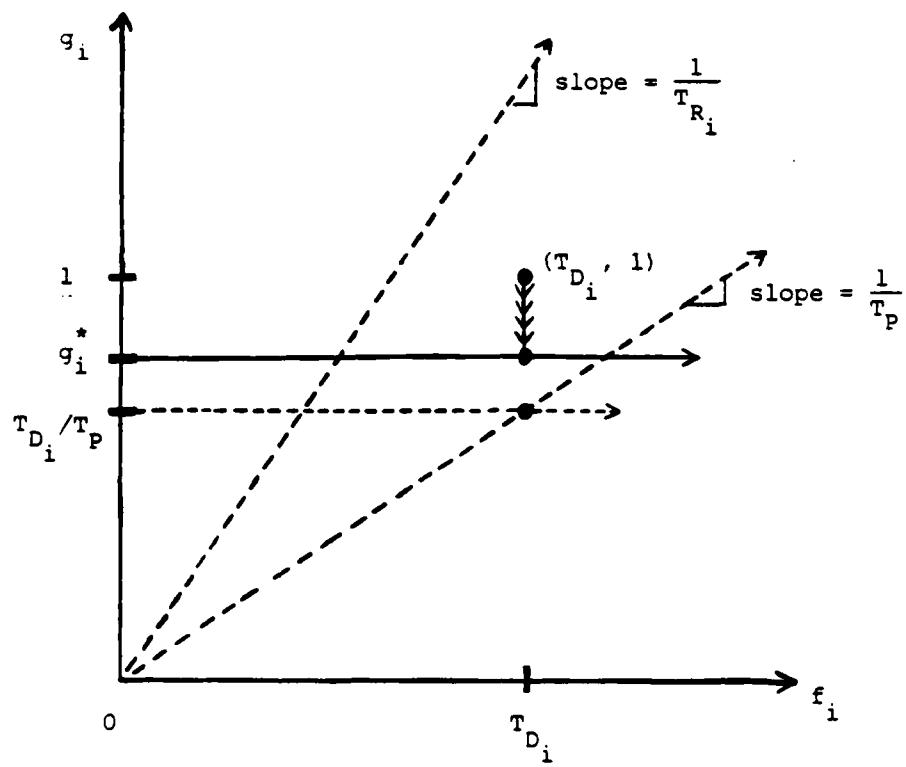
Fig. 11: Minimum-Time Trajectory (for $\dfrac{T_{D_i}}{T_P} \leq g_i^* \leq 1$)

Using (49), we see that there is no area (along the f_i-axis) under the minimum-time trajectory, so:

$$\boxed{t_{min}(g_i^*) = 0}$$

(52).

$$\underline{\text{Case \#2}} \left(\frac{T_{R_i}}{T_P} \le g_i^* \le \frac{T_{D_i}}{T_P} \right) :$$
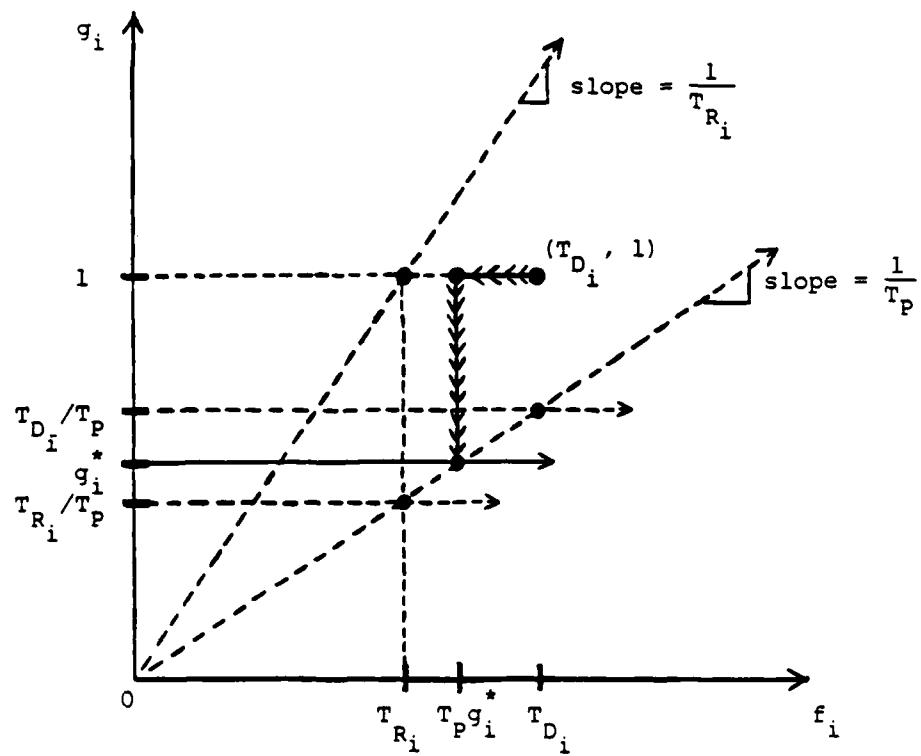
Fig. 12: Minimum-Time Trajectory (for $\frac{T_{R_i}}{T_P} \le g_i^* \le \frac{T_{D_i}}{T_P}$)

Using (49),

$$t_{min}(g_i^*) = \int_{T_P g_i^*}^{T_{D_i}} \frac{1}{(1)} \, df_i = \int_{T_P g_i^*}^{T_{D_i}} df_i = \boxed{T_{D_i} - T_P g_i^*} \qquad (53).$$

Case #3 $(0 < g_i^* \leq \dfrac{T_{R_i}}{T_P})$:
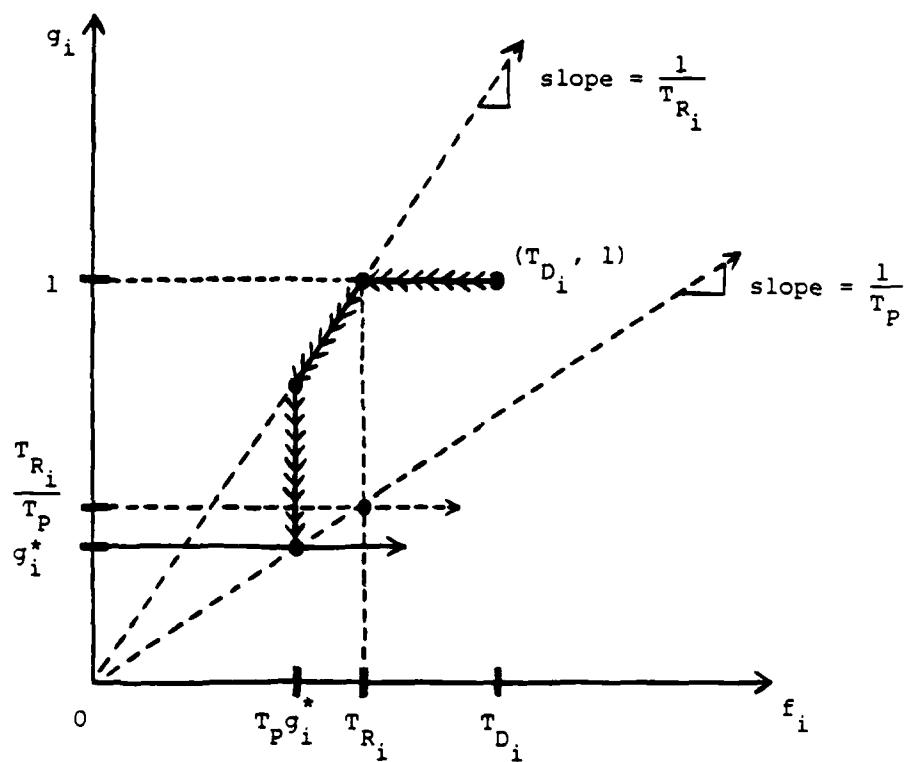

Fig. 13: Minimum-Time Trajectory (for $0 < g_i^* \leq \dfrac{T_{R_i}}{T_P}$)

Using (49),

$$t_{min}(g_i^*) = \int_{T_P g_i^*}^{T_{R_i}} \frac{1}{\left(\dfrac{f_i}{T_{R_i}}\right)} \, df_i + \int_{T_{R_i}}^{T_{D_i}} \frac{1}{(1)} \, df_i$$

$$= T_{R_i} \int_{T_P g_i^*}^{T_{R_i}} \frac{1}{f_i} \, df_i + \int_{T_{R_i}}^{T_{D_i}} df_i$$

$$= T_{R_i} [\ln(T_{R_i}) - \ln(T_P g_i^*)] + [T_{D_i} - T_{R_i}]$$

$$\boxed{= T_{R_i} \ln\left(\frac{T_{R_i}}{T_P g_i^*}\right) + T_{D_i} - T_{R_i}} \tag{54}.$$

The results in (50) - (54) can be summarized in the following graph of both $t_{max}(g_i^*)$ and $t_{min}(g_i^*)$ vs. g_i^* (over the entire range $0 < g_i^* \le 1$):
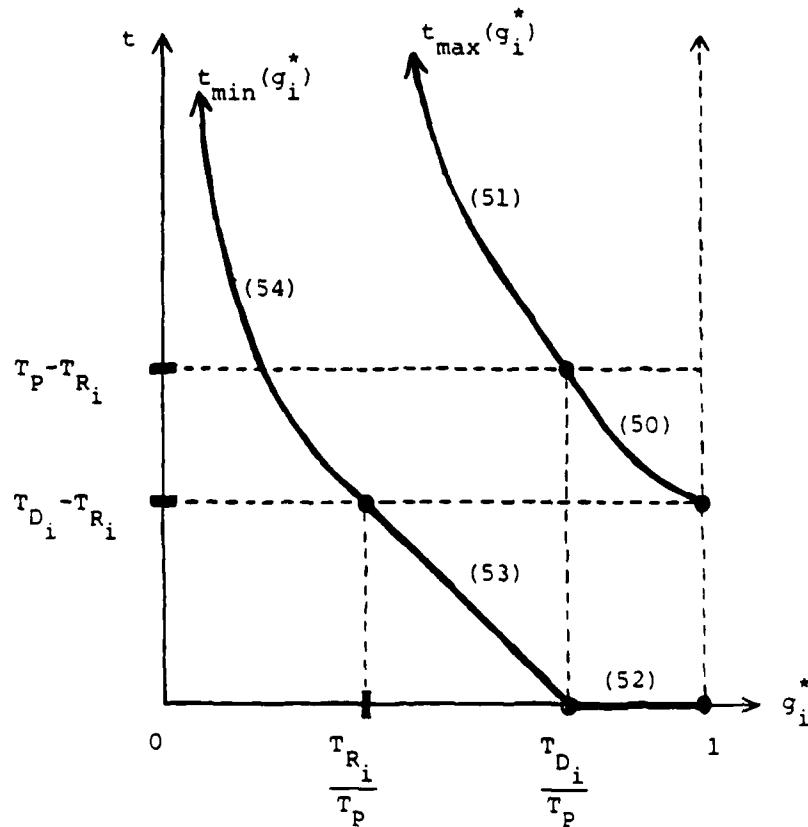
Fig. 14: Maximum and Minimum Times as a Function of g_i^*

Since $\dot{g}_i(t) \le 0$, at any instant of time t the actual value of $g_i(t)$ is less than or equal to the value of g_i^* that is reached via the maximum-time trajectory in time t, and is greater than or equal to the value of g_i^* that is reached via the minimum-time trajectory in time t. For this reason, the _inverse_ _functions_ of $t_{max}(g_i^*)$ and $t_{min}(g_i^*)$

are <u>upper</u> and <u>lower bounds</u> respectively on $g_i(t)$.

$$\underline{g}_i(t) \le g_i(t) \le \bar{g}_i(t)$$

$$\text{where} \begin{cases} \underline{g}_i(t) \stackrel{\Delta}{=} t_{min}^{-1}(t) \\ \\ \bar{g}_i(t) \stackrel{\Delta}{=} t_{max}^{-1}(t) \end{cases}$$

$$(55).$$

These bounds are plotted below:
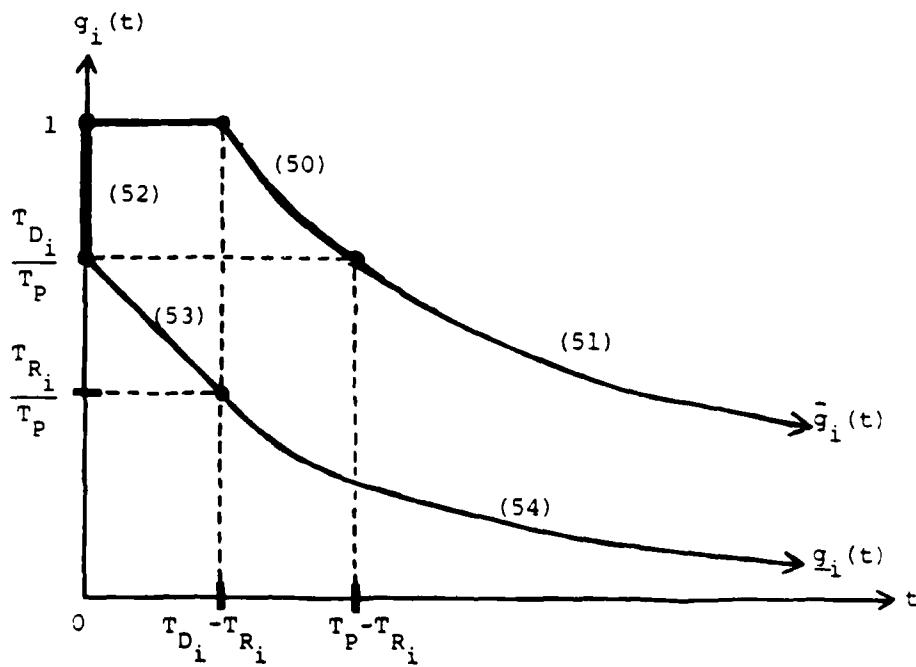
Fig. 15: Bounds on $g_i(t)$

$$\bar{g}_i(t) = \begin{cases} 1 & ; \ 0 \le t \le T_{D_i}-T_{R_i} \\ \\ \dfrac{T_{D_i}}{t+T_{R_i}} & ; \ T_{D_i}-T_{R_i} \le t \le T_P-T_{R_i} \\ \\ \dfrac{T_{D_i}}{T_P} e^{(T_P-T_{R_i})/T_P} e^{-t/T_P} & ; \ t \ge T_P-T_{R_i} \end{cases}$$

$$(56).$$

$$\underline{g}_i(t) = \begin{cases} \dfrac{T_{D_i} - t}{T_P} & ; \ 0 \leq t \leq T_{D_i} - T_{R_i} \\[2em] \dfrac{T_{R_i}}{T_P} e^{(T_{D_i} - T_{R_i})/T_{R_i}} e^{-t/T_{R_i}} & ; \ t \geq T_{D_i} - T_{R_i} \end{cases} \qquad (57).$$

Finally, the bounds on $g_i(t)$ can be easily converted to bounds on the voltage step response $v_i(t)$ as follows:

$$\underline{g}_i(t) \leq \left(g_i(t) \ \triangleq \ \frac{v_{i_{eq}} - v_i(t)}{v_{i_{eq}} - v_{i_o}} \right) \leq \bar{g}_i(t) \qquad (58).$$

$$\underline{v}_i(t) \leq v_i(t) \leq \bar{v}_i(t)$$

$$\text{where,} \begin{cases} \underline{v}_i(t) \ \triangleq \ v_{i_{eq}} - (v_{i_{eq}} - v_{i_o}) \bar{g}_i(t) \\[1em] \bar{v}_i(t) \ \triangleq \ v_{i_{eq}} - (v_{i_{eq}} - v_{i_o}) \, \underline{g}_i(t) \end{cases} \qquad (59).$$

These bounds are plotted below:
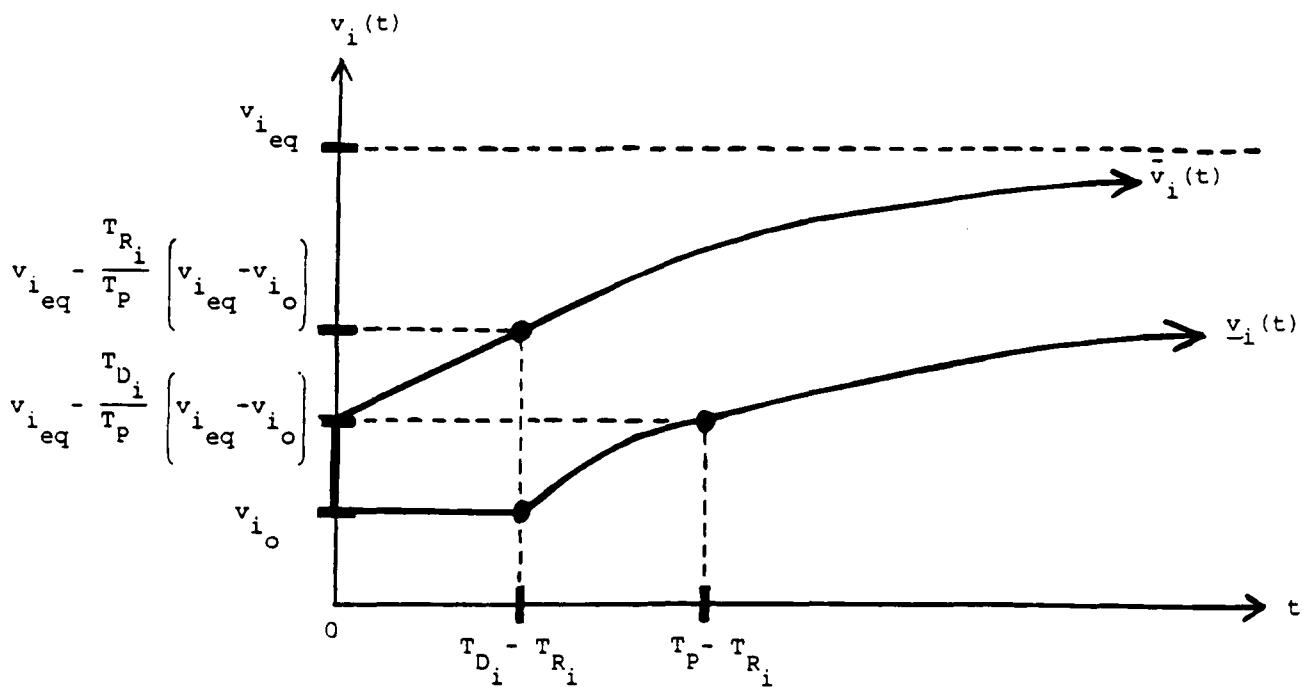
Fig. 16: Bounds on $v_i(t)$

Now that bounds for the step response have been derived, it is worthwhile to reconsider the Elmore waveform estimate discussed in Section V, $\hat{v}_i(t) \stackrel{\Delta}{=} v_{i_{eq}} + (v_{i_o} - v_{i_{eq}})e^{-t/T_{D_i}}$. When plotted in the f_i-g_i plane, the trajectory corresponding to $\hat{v}_i(t)$ is a <u>straight</u> <u>line</u> from the initial condition $(f_i(0), g_i(0)) = (T_{D_i}, 1)$ to the origin. It is therefore a <u>feasible</u> (lies inside the state-constraint cone: $\frac{f_i}{T_p} \leq g_i \leq \frac{f_i}{T_{R_i}}$) but <u>not</u> <u>optimal</u> solution to the optimal control problem. Hence <u>the Elmore waveform estimate will always lie between the upper and lower bounds</u> on the exact response $v_i(t)$,

$$\underline{v}_i(t) \leq \hat{v}_i(t) \leq \bar{v}_i(t) \tag{60}.$$

For any leaky RC mesh it is always true that $T_{R_i} \leq T_{D_i} \leq T_P$ (evaluate
the state constraints (46) at time t=0; see also Appendix B). The Elmore
waveform estimate, $\hat{v}_i(t)$, represents an effort to approximate the dynamics
of a higher order network by one with a single time constant T_{D_i} : the
estimate is exact only in that case. However, $\hat{v}_i(t)$ is a very good
estimate whenever $T_P - T_{R_i} << T_{D_i}$ because this means the state-constraint
cone shown in Fig. 8-13 is very narrow, so the actual trajectory in the
f_i-g_i plane is well-approximated by the Elmore trajectory of a straight
line of slope $\frac{1}{T_{D_i}}$. [In fact, the voltage bounds, $\underline{v}_i(t)$ and $\bar{v}_i(t)$, are
very tight whenever $T_P - T_{R_i} << T_{D_i}$ for exactly the same reason. The
minimum- and maximum-time trajectories are very close together simply by
virtue of the fact that they each must lie inside a very narrow state-
constraint cone.]

VII. Bounds for Non-Step Inputs

In the previous section we derived rigorous upper and lower
bounds on the step response voltage waveform at any selected output node
(i). A question which naturally arises is: How do we bound the response
at node (i) to an arbitrary non-step input? In this section, we will show
a general method for deriving bounds for an arbitrary input in terms of
the step response bounds. We will first consider the case where the
leaky RC mesh is at zero initial equilibrium (this treatment is essentially
the same as the one given in Appendix E of reference [1]). We will then
show how these results can be easily adapted to an arbitrary (possibly
non-zero) initial equilibrium. Finally, we will carry out the calculations

explicitly for a particular example: a <u>saturating ramp</u> input.

a.) <u>Non-Step Input (Zero Initial Equilibrium)</u>:

The method presented here depends on the fact that the RC network is <u>linear</u> (and time-invariant). We will first introduce some notation:

$h_i(t) \triangleq$ <u>zero-state</u> response at node $\textcircled{i}$ to a <u>unit</u> impulse input.

$w_i(t) \triangleq$ <u>zero-state</u> response at node $\textcircled{i}$ to a <u>unit</u> step input; so

$\dot{w}_i(t) = h_i(t)$.

[Here we use the notation $w_i(t)$ rather than $v_i(t)$. The reason for this is that $w_i(t)$ denotes a <u>more specific</u> waveform than was denoted by our previous usage of $v_i(t)$. We had previously used $v_i(t)$ to represent the response to a step input of <u>arbitrary magnitude</u> starting from some <u>arbitrary initial equilibrium</u>. Note also the difference in dimensions: $v_i(t)$ has dimensions of volts, but $w_i(t)$ is <u>dimensionless</u> since it represents the <u>unit</u> step response.]

$e(t) \triangleq$ arbitrary non-step input.

$y_i(t) \triangleq$ <u>zero-state</u> response at node $\textcircled{i}$ to the input $e(t)$.

Using Laplace transforms,

$$\left\{ \begin{array}{ll} W_i(s) = H_i(s) \cdot \dfrac{1}{s} & (61). \\[2ex] Y_i(s) = H_i(s) \cdot E(s) & (62). \end{array} \right.$$

Combining (61) and (62) we have,

$$Y_i(s) = [\tfrac{1}{s}H_i(s)] \, [sE(s)] = [W_i(s)] \, [sE(s)] \qquad (63).$$

But, $sE(s) = L\{\dfrac{de(t)}{dt}\}$; assuming $e(0)=0$ \qquad (64).

So converting (63) back to the time domain (and using (64)), we have:

$$y_i(t) = w_i(t) * \frac{de(t)}{dt} = \int_0^t w_i(t-t') \frac{de(t')}{dt} \, dt' \qquad (65).$$

The step response bounds for $w_i(t)$ are given by:

$$\underline{w}_i(t) \leq w_i(t) \leq \bar{w}_i(t) \qquad (66),$$

where (referring back to (59)) we have:

$$\begin{cases} \underline{w}_i(t) = x_i(1-\bar{g}_i(t)) \\[2mm] \bar{w}_i(t) = x_i(1-\underline{g}_i(t)) \end{cases} \qquad (67),$$

since $w_{i_o} = 0$ and $w_{i_{eq}} = x_i$.

In order to obtain bounds on $y_i(t)$, we note that convolution of (66) with $\frac{de(t)}{dt}$ retains the sense of the inequality when $\frac{de(t)}{dt} \geq 0 \; \forall t \geq 0$, and reverses the sense of the inequality when $\frac{de(t)}{dt} \leq 0 \; \forall t \geq 0$, i.e.,

$$\begin{cases} \underline{w}_i(t) * \frac{de(t)}{dt} \leq \left(y_i(t) = w_i(t) * \frac{de(t)}{dt}\right) \leq \bar{w}_i(t) * \frac{de(t)}{dt} \; ; \; \frac{de(t)}{dt} \geq 0 \; \forall t \geq 0 \\[3mm] \bar{w}_i(t) * \frac{de(t)}{dt} \leq \left(y_i(t) = w_i(t) * \frac{de(t)}{dt}\right) \leq \underline{w}_i(t) * \frac{de(t)}{dt} \; ; \; \frac{de(t)}{dt} \leq 0 \; \forall t \geq 0 \end{cases}$$

$$(68).$$

So, when the arbitrary input $e(t)$ is <u>monotone</u>, the bounds on $y_i(t)$ are given directly by (68):

$$\underline{y}_i(t) \leq y_i(t) \leq \bar{y}_i(t) \qquad (69).$$

$$\underline{y}_i(t) = \begin{cases} \displaystyle\int_0^t \underline{w}_i(t-t') \, \frac{de(t')}{dt'} \quad ; \ e(t) \text{ increasing} \\[3em] \displaystyle\int_0^t \bar{w}_i(t-t') \, \frac{de(t')}{dt'} \quad ; \ e(t) \text{ decreasing} \end{cases}$$

$$\bar{y}_i(t) = \begin{cases} \displaystyle\int_0^t \bar{w}_i(t-t') \, \frac{de(t')}{dt'} \, dt' \ ; \ e(t) \text{ increasing} \\[3em] \displaystyle\int_0^t \underline{w}_i(t-t') \, \frac{de(t')}{dt'} \, dt' \ ; \ e(t) \text{ decreasing} \end{cases}$$

In typical CAD applications, the input $e(t)$ is modelled by a monotone waveform. Hence the bounds given in (69) are sufficient for most cases. However in the rare case where the input $e(t)$ has both positive and negative slopes, we can still bound the response $y_i(t)$ in a similar way. We do so by considering separately the time intervals where $\frac{de(t)}{dt} > 0$ and $\frac{de(t)}{dt} < 0$ when we perform the convolution operation (for details, see Appendix E of [1]).

b.) Non-Step Input (Arbitrary Initial Equilibrium)

Suppose that we desire bounds on the response $y_i(t)$ to a non-step input $e(t)$ when we are starting from some non-zero equilibrium.

Define:

$$\begin{cases} \Delta e(t) \triangleq e(t) - e(0) \\[1em] \Delta \underline{y}(t) \triangleq \underline{y}(t) - \underline{y}(0) \quad [\Delta y_i(t) \triangleq y_i(t) - y_i(0)] \end{cases} \qquad (70).$$

We can derive bounds for the response $\Delta y_i(t)$ to the non-step input $\Delta e(t)$ by the convolution procedure discussed in part (a), since the $(\underline{\Delta} \underline{y}, \underline{\Delta} \underline{e})$ system is <u>starting</u> <u>from</u> <u>a</u> <u>zero</u> <u>equilibrium</u>. Just add $y_i(0)$ to these bounds in order to obtain bounds on $y_i(t)$:

$$\underline{y}_i(t) \leq y_i(t) \leq \bar{y}_i(t)$$

$$\text{where,} \begin{cases} \underline{y}_i(t) = y_i(0) + \underline{\Delta y}_i(t) \\ \bar{y}_i(t) = y_i(0) + \overline{\Delta y}_i(t) \end{cases} \tag{71}.$$

The justification for this is that the $(\Delta \underline{y}, \Delta \underline{e})$ system is described by the same differential equation as the $(\underline{y}, \underline{e})$ system:

$$\Delta^{\cdot} \underline{y}(t) = \dot{\underline{y}}(t)$$

$$= [RC]^{-1} (\underline{\underline{x}} \underline{e}(t) - \underline{y}(t)) \; ; \; \text{see (9)}$$

$$\overset{\Delta}{=} [RC]^{-1} (\underline{\underline{x}} (\Delta\underline{e}(t) + \underline{e}(0)) - (\Delta \underline{y}(t) + \underline{y}(0)))$$

$$= [RC]^{-1} (\underline{\underline{x}} \Delta \underline{e}(t) + \underline{\underline{x}}\underline{e}(0) - \Delta \underline{y}(t) - \underline{y}(0))$$

$$\uparrow \underline{\hspace{3cm}} \uparrow$$

cancel, since $\underline{y}(0) = \underline{\underline{x}}\underline{e}(0)$

$$= [RC]^{-1} (\underline{\underline{x}} \Delta \underline{e}(t) - \Delta \underline{y}(t)) \tag{72}.$$

c.) <u>Example: Saturating Ramp Input</u>

The non-step input, $e(t)$, considered in this example will be called a "saturating ramp" since it starts from zero volts at time $t=0$

and rises linearly until it saturates at V_{sat} volts at time $t = T_{sat}$ (see Fig. 17).
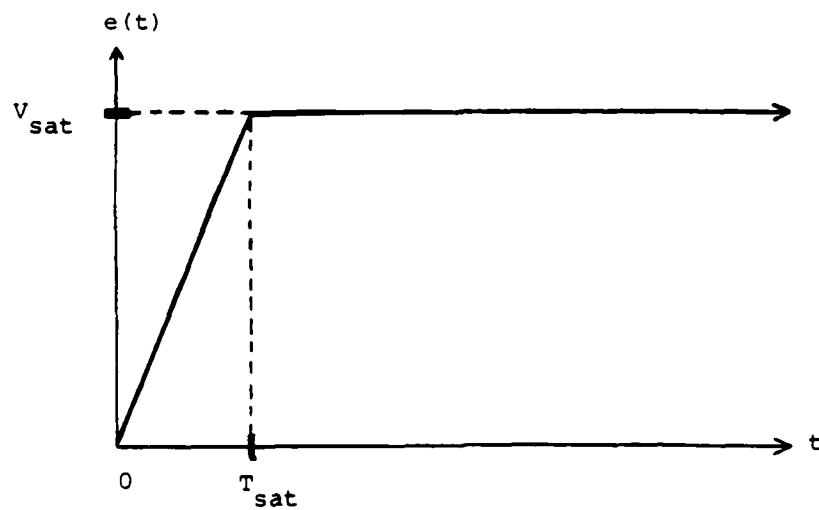

Fig. 17: Saturating Ramp Input

Furthermore, in this example the bounds calculations will be done under the assumption that the ramp rises sharply compared to the characteristic time constants of the RC network being driven. Specifically, we shall assume:

$$0 < T_{sat} \leq \min \{ (T_{D_i} - T_{R_i}), (T_P - T_{D_i}) \}$$ (73).

[There is nothing fundamental about the assumption in (73). Upper and lower bounds for _any_ choice of T_{sat} can be calculated by the procedure outlined in this section; the only difference being that the convolution integrals in (69) could assume a different form.]

In the following bounds calculations for the zero-state response at node (i) to the saturating ramp input $e(t)$, we will make use of the following abbreviation:

$$K_i \triangleq \frac{V_{sat}}{T_{sat}} x_i \qquad (74).$$

(Note that x_i is dimensionless, so that K_i has dimensions of volts/seconds.)

Upper Bound: There are four different upper bound formulas depending on the value of time t.

Using (69), we have:

1) for $0 < t \leq T_{sat}$,

$$\bar{y}_i(t) = K_i \int_0^t [1 - \frac{T_{D_i} - (t-t')}{T_P}] dt'$$

$$= \boxed{K_i [(1 - \frac{T_{D_i}}{T_P})t + \frac{t^2}{2T_P}]} \qquad (75).$$

2) for $T_{sat} \leq t \leq T_{D_i} - T_{R_i}$,

$$\bar{y}_i(t) = K_i \int_0^{T_{sat}} [1 - \frac{T_{D_i} - (t-t')}{T_P}] dt'$$

$$= \boxed{K_i [(1 - \frac{T_{D_i}}{T_P} - \frac{T_{sat}}{2T_P}) T_{sat} + \frac{T_{sat}}{T_P} t]} \qquad (76).$$

3) for $T_{D_i} - T_{R_i} \leq t \leq (T_{D_i} - T_{R_i}) + T_{sat}$,

$$\bar{y}_i(t) = K_i \int_0^{t-(T_{D_i}-T_{R_i})} [1 - \frac{T_{R_i}}{T_P} e^{(T_{D_i}-T_{R_i})/T_{R_i}} e^{-(t-t')/T_{R_i}}] dt'$$

$$+ K_i \int_{t-(T_{D_i}-T_{R_i})}^{T_{sat}} [1 - \frac{T_{D_i} - (t-t')}{T_P}] dt'$$

$$= \left| K_i \left\{ \frac{1}{T_p} (T_{D_i} - t)[t - (T_{D_i} - T_{R_i})] + \frac{1}{2T_p} [t - (T_{D_i} - T_{R_i})]^2 \right. \right.$$

$$\left. \left. - \frac{T_{R_i}^2}{2T_p} [1 - e^{-(t-(T_{D_i}-T_{R_i}))/T_{R_i}}] + (1 - \frac{T_{D_i}}{T_p} - \frac{T_{sat}}{2T_p})T_{sat} + \frac{T_{sat}}{T_p} t \right\} \right|$$

$$(77).$$

4) $\underline{\text{for } t \geq (T_{D_i} - T_{R_i}) + T_{sat}}$,

$$\bar{y}_i(t) = K_i \int_0^{T_{sat}} [1 - \frac{T_{R_i}}{T_p} e^{(T_{D_i}-T_{R_i})/T_{R_i}} e^{-(t-t')/T_{R_i}}] dt'$$

$$= \left| K_i \left\{ T_{sat} - \frac{T_{R_i}^2}{T_p} e^{-(t-(T_{D_i}-T_{R_i}+T_{sat}))/T_{R_i}} [1 - e^{-T_{sat}/T_{R_i}}] \right\} \right|$$

$$(78).$$

Lower Bound: There are five different lower bound formulas depending on the value of time t.

Using (69), we have:

1) $\underline{\text{for } 0 \leq t \leq T_{D_i} - T_{R_i}}$,

$$\boxed{\underline{y}_i(t) = 0} \qquad\qquad (79).$$

2) $\underline{\text{for } T_{D_i} - T_{R_i} \leq t \leq (T_{D_i} - T_{R_i}) + T_{sat}}$,

$$\underline{y}_i(t) = K_i \int_0^{t-(T_{D_i}-T_{R_i})} [1 - \frac{T_{D_i}}{(t-t')+T_{R_i}}] dt'$$

$$= \left| K_i \left\{ [t - (T_{D_i} - T_{R_i})] - T_{D_i} \ln \left(\frac{t+T_{R_i}}{T_{D_i}} \right) \right\} \right| \qquad (80).$$

3) for $(T_{D_i} - T_{R_i}) + T_{sat} \leq t \leq (T_P - T_{R_i})$,

$$\underline{Y}_i(t) = K_i \int_0^{T_{sat}} [1 - \frac{T_{D_i}}{(t-t')+T_{R_i}}] \, dt'$$

$$= \boxed{K_i \left\{ T_{sat} - T_{D_i} \ln \left(\frac{t+T_{R_i}}{t+T_{R_i}-T_{sat}} \right) \right\}} \qquad (81).$$

4) for $T_P - T_{R_i} \leq t \leq (T_P - T_{R_i}) + T_{sat}$,

$$\underline{Y}_i(t) = K_i \int_0^{t-(T_P-T_{R_i})} [1 - \frac{T_{D_i}}{T_P} e^{(T_P-T_{R_i})/T_P} e^{-(t-t')/T_P}] \, dt'$$

$$+ K_i \int_{t-(T_P-T_{R_i})}^{T_{sat}} [1 - \frac{T_{D_i}}{(t-t')+T_{R_i}}] \, dt'$$

$$= \boxed{K_i \left\{ T_{sat} - T_{D_i} \ln \left(\frac{T_P}{t+T_{R_i}-T_{sat}} \right) - T_{D_i} [1 - e^{-(t-(T_P-T_{R_i}))/T_P}] \right\}}$$

$$(82).$$

5) for $t \geq (T_P - T_{R_i}) + T_{sat}$,

$$\underline{Y}_i(t) = K_i \int_0^{T_{sat}} [1 - \frac{T_{D_i}}{T_P} e^{(T_P-T_{R_i})/T_P} e^{-(t-t')/T_P}] \, dt'$$

$$= \boxed{K_i \left\{ T_{sat} - T_{D_i} e^{-(t-(T_P-T_{R_i}+T_{sat}))/T_P} [1 - e^{-T_{sat}/T_P}] \right\}}$$

$$(83).$$

VIII. Additional Computational Burdens Arising from the Presence of Resistive Paths to Ground

The formulas for the time constants T_{R_i}, T_{D_i}, and T_p indicate that the information required in order to compute bounds for leaky RC meshes is as follows:

1) The ith row (or column) of the resistance matrix R.

2) The diagonal elements of the resistance matrix R.

3) All elements of the vector x (i.e., the equilibrium voltage distribution for the entire network when a 1-volt source is applied at the input).

Items 1) and 2) above do not represent much of an additional burden over non-leaky RC meshes that happen to contain resistor loops. If a non-leaky RC mesh contains resistor loops, it is already difficult to calculate the entries of the R matrix. The presence of grounded resistors does not make this already difficult problem that much harder. If, however, the RC network under consideration is a leaky RC tree, than items 1) and 2) post a clear-cut extra computational burden. The reason for this is that the entries of the R matrix will require some computation: we cannot read them off by inspection as we can with a non-leaky RC tree [1].

Item 3) above represents a clear-cut extra computational burden regardless of what the underlying non-leaky RC mesh looks like. For a non-leaky RC mesh, all equilibrium voltages are equal. If resistive paths to ground are present, then some computation is required. However, as the following lemma shows, the extra computation for the equilibrium

voltage distribution caused by the presence of <u>any number</u> of grounded
resistors in the RC network is no worse than the extra computation required
to consider <u>only one</u> (in most cases) extra node in the RC network as an
output node to be bounded.

<u>Lemma</u>: Assume (as is usually the case) that the RC network under
consideration has only one capacitor node connected to the voltage
source through a single resistor (see Fig. 18). Label this node
as "①", and label the resistor connecting the source to node ①
as "R_s" ("source resistance"). We have the following relation:

$$x_k = \frac{R_{kl}}{R_s} \quad ; \; \forall \text{ nodes k in the RC network} \qquad (84),$$

where R_{kl} is an element of the resistance matrix R.
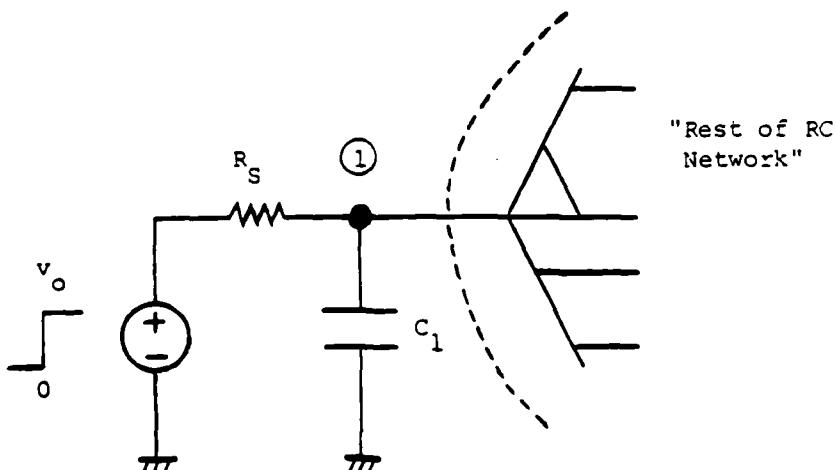

Fig. 18: RC network with only one node connected
directly to the source.

<u>Proof of Lemma:</u>

Assume the input voltage source is a step of magnitude v_o volts at time t=0, and that the network is starting from a zero equilibrium condition. The differential equation for the RC network is,

$$[RC]\dot{v}(t) = v_{eq} - v(t) \tag{85}.$$

Taking the kth row at time $t=0^+$, we have:

$$\sum_j R_{kj}C_j\dot{v}_j(0^+) = v_{k_{eq}} - v_k(0^+) \tag{86}.$$

But $v_k(0^+) = 0$ for all k, and $\dot{v}_j(0^+) = 0$ for all j <u>except</u> j=1. So (86) becomes,

$$R_{k1}C_1\dot{v}_1(0^+) = v_{k_{eq}} \tag{87}.$$

We identify $C_1\dot{v}_1(0^+) = \dfrac{v_o}{R_s}$ as the current flowing into node ① at time $t=0^+$,

$$R_{k1} \cdot \frac{v_o}{R_s} = v_{k_{eq}} \tag{88},$$

$$\Rightarrow x_k = \frac{v_{k_{eq}}}{v_o} = \frac{R_{k1}}{R_s} \tag{89}.$$

This proves the lemma. ■

This lemma shows that computing x_k for all k is equivalent to computing R_{k1} for all k [i.e., the first column (or row) of the resistance matrix R] However, the first column (or row) of R is exactly what would be needed if we wanted to bound the response at node ①. Hence, the calculation of the equilibrium voltage distribution is no worse computationally

than considering one more node as an output node.

Note that in the rare case when there are M>1 capacitor nodes connected directly to the source by resistors $R_{S_1}, \ldots, R_{S_M}$, then a straightforward extension to the preceding lemma holds:

$$x_k = \sum_{j=1}^{M} \frac{R_{kj}}{R_{S_j}} \; ; \; \forall \text{ nodes } k \tag{90}.$$

This means that calculation of the equilibrium voltage distribution is equivalent to computing the first M columns (or rows) of R, which is computationally equivalent to considering nodes ①,....⑩ as additional output nodes to be bounded.

The computational problem now consists of finding the first row (or column) of R, the ith row (or column) of R for each output node ⓘ, and the diagonal elements of R. Unfortunately, the R matrix is "global" in character and is difficult to compute if there are resistor loops or resistive paths to ground. However, the conductance matrix $G = R^{-1}$ is "local" in character, sparse (i.e., has many zero entries), and can be determined virtually by inspection even when there are resistor loops or grounded resistors present. So one approach is to determine G directly and then invert it to find R. This approach, however, raises some doubts because it finds all of R instead of just the entries of R that are actually needed. An alternate approach is given below.

Alternate Approach:

In this alternate approach, we only calculate the entries of the R matrix that are needed. If the entry R_{ik} of R is desired, the conductance

matrix is "personalized" to nodes ⓘ and ⓚ by <u>eliminating</u> nodes in the
original resistive subnetwork as much as possible without destroying the
topology of node ⓘ with respect to node ⓚ. We will illustrate this
method with an example calculation.
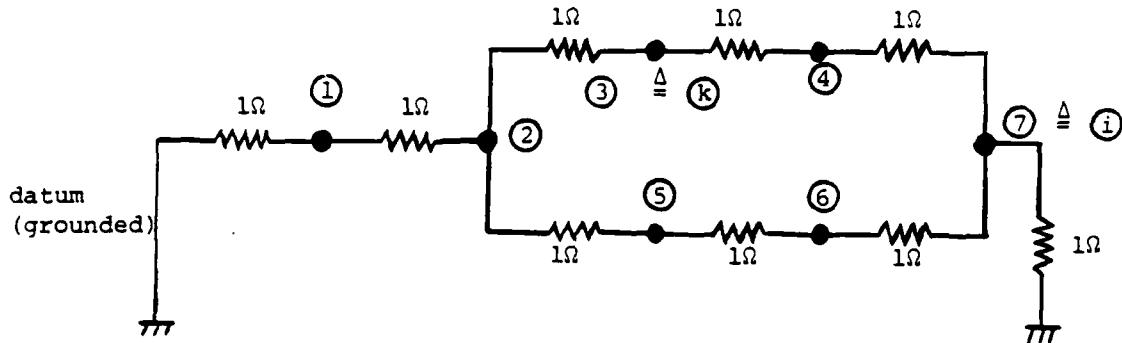
Fig. 19:   Resistive Subnetwork for Example Calculation of
R_{ik} (all resistors are 1-ohm)

The resistive subnetwork for the example calculation is shown in Fig. 19.
We wish to calculate the entry R_{ik} of the R matrix that describes this
resistor subnetwork. The first step is to "personalize" the subnetwork
to nodes ⓘ and ⓚ by eliminating as many nodes as possible (see Fig. 20).
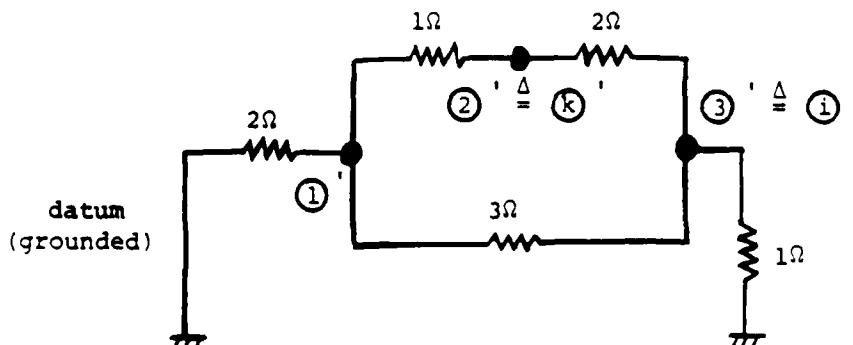
Fig. 20:   Resistive Subnetwork from Fig. 19,
Personalized to Nodes ⓘ and ⓚ

<u>Note</u>: We will use a prime to distinguish the subnetwork in Fig. 20 from the subnetwork in Fig. 19.

The key point to notice about this node elimination procedure is that,

$$\boxed{R'_{ik} = R_{ik}} \tag{91}.$$

Hence, we can determine R_{ik} from the <u>much smaller</u> conductance matrix $\underset{\sim}{G}'$ (rather than $\underset{\sim}{G}$), either by directly inverting $\underset{\sim}{G}'$ or by using cofactors:

$$R_{ik} = R'_{ik} = R'_{32} = [(\text{cof } G'_{23})/(\det \underset{\sim}{G}')] \tag{92}.$$

The conductance matrix $\underset{\sim}{G}'$ can be read off quickly from Fig. 20 as:

$$\underset{\sim}{G}' = \begin{bmatrix} \frac{11}{6} & -1 & -\frac{1}{3} \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{11}{6} \end{bmatrix} \tag{93}.$$

Using the cofactor method, we have:

$$\det \underset{\sim}{G}' = \frac{11}{6} \begin{vmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{11}{6} \end{vmatrix} + \begin{vmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{3} & \frac{11}{6} \end{vmatrix} - \frac{1}{3} \begin{vmatrix} -1 & \frac{3}{2} \\ -\frac{1}{3} & -\frac{1}{2} \end{vmatrix} = \frac{9}{4} \tag{94}.$$

$$\text{cof } G'_{23} = - \begin{vmatrix} \frac{11}{6} & -1 \\ -\frac{1}{3} & -\frac{1}{2} \end{vmatrix} = \frac{5}{4} \tag{95}.$$

$$R_{ik} = R'_{32} = \frac{(5/4)}{(9/4)} = \frac{5}{9} \qquad (96).$$

This result of $R_{ik} = \frac{5}{9} \, \Omega$ is correct, and can be verified by going back to Fig. 19 without considering Fig. 20 at all.

In summary, this approach suggests a tradeoff. Instead of doing one calculation with the large original conductance matrix $\underset{\sim}{G}$, you can do several calculations with smaller "personalized" conductance matrices $\underset{\sim}{G}'$.

Acknowledgement:

Appendix A: Proof of Facts 1 and 2

The proof of Fact 1 depends on additional structure of the conductance matrix G alluded to in Section III.

Lemma: Assuming all resistors in a given resistor subnetwork R are positive, the conductance matrix G which describes R has the following properties:

1) $G_{nn} > 0$ $\forall n$; (i.e., all diagonal elements are positive)

2) $G_{mn} \leq 0$ $\forall m \neq n$; (i.e., all off-diagonal elements are non-positive)

Proof of Lemma:

Apply a 1-volt source to node $\textcircled{n}$ of the resistor subnetwork R. Ground the datum and all external nodes other than $\textcircled{n}$.

The Thevenin equivalent circuit looking into node $\textcircled{n}$ is a positive resistor to ground ($v_{thev}=0$). Since we are applying a 1-volt source to node $\textcircled{n}$, positive current flows <u>into</u> node $\textcircled{n}$ $\Rightarrow G_{nn} > 0$.

The Thevenin equivalent circuit looking into any other node $\textcircled{m}$ ($m \neq n$) is either a positive resistor to ground ($v_{thev}=0$; occurs when node $\textcircled{m}$ is not connected to node $\textcircled{n}$ through a single resistor), or is a positive resistor in series with a positive* voltage source ($v_{thev} > 0$; occurs when node $\textcircled{m}$ is connected to node $\textcircled{n}$ through a single resistor).

---

*It is intuitively clear that the open-circuit voltage at node $\textcircled{m}$ is non-negative. This fact is a special case of the "voltage minimax theorem" [3; pg. 778].

In either case, since we are applying ground to node (m) , there can be no positive current flowing <u>into</u> node (m) $\Rightarrow G_{mn} \leq 0$. ∎

Proof of Fact 1:

(See Section VI for the statement of Fact 1.)

The proof below is essentially the same as for non-leaky RC meshes given in [2]:

If i=j or i=k, (24) is satisfied trivially. If j=k, then (24) becomes $R_{ii}R_{kk} \geq R_{ki}^2$, since $\underset{\sim}{R}$ is symmetric, and is true because $\underset{\sim}{R}$ is positive-definite and hence the 2x2 principal submatrix of $\underset{\sim}{R}$ consisting of elements in rows i or k that also lie in columns i or k must have a positive determinant. Now suppose i, j, and k are distinct, and consider the resistive subnetwork $\hat{R}$ obtained from the original resistive sub-network R by open-circuiting all external terminals of R except (i) , (j) (k) , and the datum. Taking terminals (i), (j), (k) of R as terminals (1), (2), (3) respectively of $\hat{R}$, we see that $\hat{R}$ has the following resistance matrix:

$$\hat{\underset{\sim}{R}} = \begin{bmatrix} R_{ii} & R_{ij} & R_{ik} \\ R_{ji} & R_{jj} & R_{jk} \\ R_{ki} & R_{kj} & R_{kk} \end{bmatrix} \tag{A-1}.$$

Now consider $\hat{\underset{\sim}{G}} = \hat{\underset{\sim}{R}}^{-1}$, the conductance matrix of $\hat{R}$. The off-diagonal elements of $\hat{\underset{\sim}{G}}$ are non-positive (by the preceding lemma), so in particular $\hat{G}_{kj} \leq 0$. Taking the inverse of $\hat{R}$ using cofactors, we see that $\hat{G}_{kj} = (\hat{\underset{\sim}{R}}^{-1})_{kj} = [(\text{cof } R_{jk})/(\det \hat{\underset{\sim}{R}})] \leq 0$. Since $\hat{R}$ is positive-definite

$\Longrightarrow \det \hat{\underline{R}} > 0$, it follows that cof $R_{jk} \leq 0$. But cof $R_{jk} = (-1)^{2+3}(R_{ii}R_{kj}-R_{ki}R_{ij})$. This proves (24).

Proof of Fact 2:

(See Section VI for the statement of Fact 2.)

Assume that the step transition is "up", i.e., $\underline{v}_{eq} > \underline{v}_o$. We want to prove that $\dot{v}_j(t) \geq 0$, $\forall$ nodes j in the leaky RC mesh. (The argument for a "down" step is entirely analogous.)

Note first that $\dot{v}_j(0^+) > 0$ for every capacitor connected to the source through a single resistor, and $\dot{v}_j(0^+) = 0$ for all other capacitors. Therefore $\dot{\underline{v}}(0^+)$ lies in the "first orthant" of $\mathbb{R}^N$ (the generalization to N-dimensions of the "first quadrant" of $\mathbb{R}^2$).

We claim that $\dot{\underline{v}}(t)$, once inside the first orthant, can never leave. To see this, first differentiate (12) to obtain the following differential equation for the evolution of $\dot{\underline{v}}(t)$:

$$\ddot{\underline{v}}(t) = -[\underline{RC}]^{-1} \dot{\underline{v}}(t) = -[\underline{C}^{-1}\underline{G}] \dot{\underline{v}}(t) \qquad (A-2).$$

The off-diagonal elements of G are non-positive (by the previous lemma), so the off-diagonal elements of $-[\underline{C}^{-1}\underline{G}]$ are non-negative, i.e., $-[\underline{C}^{-1}\underline{G}]$ is a "quasi-monotone increasing" operator [10]. If $\dot{\underline{v}}(t)$ were to exit the first orthant, it would have to do so by passing through the boundary, i.e., one of the axis planes. Thus there would exist an instant t^* and a capacitor k such that $\dot{v}_k(t^*) = 0$ although $\dot{v}_j(t^*) \geq 0 \ \forall j \neq k$. But from (A-2), this implies:

$$\ddot{v}_k(t^*) = - \sum_{j \neq k} \frac{G_{kj}}{C_k} \dot{v}_j(t^*) \geq 0 \qquad (A-3).$$

So in fact $\dot{v}_k(t)$ does not become negative at $t=t^*$. In other words, the vector field (A-2) never points out of the first orthant when evaluated on its boundary. (For a "down" step, $\dot{v}(0^+)$ lies in the negative "first orthant" and can never leave it.)

For a more complete explanation, and consideration of limiting cases, see references [11], [12], and [13].

Appendix B: Term-by-Term Ordering of the Time Constants

We have the following formulas for the time constants,

$$
\left\{
\begin{array}{l}
T_{R_i} = \dfrac{1}{R_{ii}} \sum_k R_{ik}^2 C_k \\[2em]
T_{D_i} = \dfrac{1}{x_i} \sum_k R_{ik} C_k x_k \\[2em]
T_P = \sum_k R_{kk} C_k
\end{array}
\right.
\qquad \text{(B-1)}.
$$

It is already known that the inequality $T_{R_i} \leq T_{D_i} \leq T_P$ holds for any leaky RC mesh (just evaluate the state constraints (46) at time t=0). However, it is not immediately obvious that this inequality also holds term-by-term for each corresponding term in the summation (i.e.,

$\dfrac{1}{R_{ii}} R_{ik}^2 C_k \leq \dfrac{1}{x_i} R_{ik} C_k x_k \leq R_{kk} C_k$ for each node k). The goal here is to establish this result. After simplification, the inequality we wish to prove reduces to:

$$
\frac{R_{ik}}{R_{ii}} \leq \frac{x_k}{x_i} \leq \frac{R_{kk}}{R_{ik}}
\qquad \text{(B-2)}.
$$

Proof of (B-2):

Starting from zero initial equilibrium, apply an "up" step of 1 volt at time t=0 to the input of any leaky RC mesh. Note that $v_o = 0$ and $v_{eq} = x$. Evaluating (26) at time t=0, we obtain:

$$R_{ii}[x_k - 0] \geq R_{ki}[x_i - 0] \implies \frac{x_k}{x_i} \geq \frac{R_{ki}}{R_{ii}} \qquad \text{(B-3)}.$$

Evaluating (27) at time t=0, we obtain:

$$R_{ki}[x_k - 0] \leq R_{kk}[x_i - 0] \implies \frac{x_k}{x_i} \leq \frac{R_{kk}}{R_{ki}} \qquad \text{(B-4)}$$

The fact that $\underline{R}$ is symmetric (so that $R_{ki} = R_{ik}$) together with (B-3) and (B-4) proves (B-2). ■

Appendix C: M-Matrix Systems

The purpose of this appendix is to outline (from a more abstract "systems" point-of-view) the underlying reason why the waveform bounding results presented in [1] and [2] carry over to the leaky RC meshes in this report. Leaky RC meshes are similar to non-leaky RC meshes [2] and non-leaky RC trees [1] in one fundamental respect: They are all linear dynamical systems with an "M-matrix" as the system matrix [14]. In the literature, there are many different equivalent statements about M-matrices that would all serve equally well as definitions. The statement best suited to our purposes is as follows.

> Definition: Let M be a non-singular square matrix of real numbers, and let $P \triangleq M^{-1}$. Then M is said to be an "M-matrix" if all off-diagonal elements of M are non-positive and all elements of P are non-negative.

In terms of the RC networks in this report, the M-matrix is

$$M = C^{-1}G \qquad \text{(C-1)},$$

and the non-negative inverse M-matrix is

$$P = RC \qquad \text{(C-2)},$$

as we can easily verify.

The conductance matrix G has non-positive off-diagonal elements

(by the lemma in Appendix A). Premultiplying by the positive diagonal matrix C^{-1} does not change this, i.e., $[C^{-1}G]$ has non-positive off-diagonal elements.

The resistance matrix R is clearly non-negative, assuming that the individual resistors are non-negative. Postmultiplying by the positive diagonal matrix C does not change this, i.e., [RC] is non-negative.

Hence $[C^{-1}G]$ is indeed an M-matrix, as is G itself. (In general, the set of M-matrices is invariant under the group action consisting of right and left multiplication by positive diagonal matrices.)

The network differential equations (12) can now be viewed more abstractly as a linear dynamical system with a (negative) M-matrix as the system matrix:

$$\dot{v}(t) = -M\,v(t) + M\,v_{eq} \qquad (C-3).$$

We can successfully bound the step response of the abstract system, (C-3), because both Fact 1 (appropriately modified) and Fact 2 from Section VI apply to the abstract system.

Fact 1-C: Let P be a non-singular NxN square matrix which is the inverse of an M-matrix. Then P has the following structure:

$$P_{ii}P_{kj} \geq P_{ki}P_{ij} \text{ for any } i,j,k \in \{1,\ldots, N\} \qquad (C-4).$$

Proof of Fact 1-C:

If i=j or i=k, (C-4) is satisfied trivially. If j=k, (C-4) becomes $p_{ii}p_{kk} \geq p_{ki}p_{ik}$ (note that $p_{ki} \neq p_{ik}$ in general; the inverse M-matrix P is not necessarily symmetric), which is true because any principal minor of an inverse M-matrix is known to be positive [16; Cor. 1, p. 198] and hence the 2x2 principal submatrix of P consisting of the elements in rows i or k that also lie in columns i or k must have a positive determinant.

Suppose i, j, and k are distinct, and consider the matrix $\hat{P}$ which is the 3x3 principal submatrix of P consisting of the elements in rows i, j, or k that also lie in columns i, j, or k. Since any principal submatrix of an inverse M-matrix is known to be an inverse M-matrix [15, p. 329], $\hat{P}$ is an inverse M-matrix and hence $(\hat{P}^{-1})_{kj} \leq 0$. Taking the inverse of $\hat{P}$ using cofactors, we have $0 \geq (\hat{P}^{-1})_{kj} = [(\text{cof } p_{jk}/(\det \hat{P})]$, and det $\hat{P} > 0$ [16]. Thus cof $p_{jk} = p_{ki}p_{ij} - p_{ii}p_{kj} \leq 0$, proving (C-4). ∎

Note: Although not proven here, Fact 1-C also holds for a slightly more general class of matrices than inverse M-matrices. Going back to the more concrete domain of RC networks, if any individual resistor or capacitor is equal to zero then Fact 1-C still holds for P = RC even though P is not invertible in this case. However, P is contained in the closure of the set of inverse M-matrices. The practical advantage of this is that the bounding technique is still valid for circuits with minor specification errors (e.g., leaving out a capacitor).

> **Fact 2-C:** If $v(t)$ in the abstract system (C-3) is initially
>
> non-decreasing (non-increasing) it will remain
>
> non-decreasing (non-increasing) for all time:
>
> i.e.,
>
> $$\dot{v}(0) \geq 0 \implies \dot{v}(t) \geq 0 \quad \forall t > 0$$
>
> $$\dot{v}(0) \leq 0 \implies \dot{v}(t) \leq 0 \quad \forall t > 0 \qquad (C-5).$$

Proof of Fact 2-C:

The differential equation for the evolution of $\dot{v}(t)$, $\ddot{v}(t) = -M\,\dot{v}(t)$, is quasi-monotone increasing (since the off-diagonal elements of $-M$ are non-negative). Hence the proof is entirely analogous to the proof in Appendix A. ∎

References

[1]   Rubinstein, J., P. Penfield, Jr., and M.A. Horowotz, "Signal Delay in RC Tree Networks," _IEEE Trans. Computer-Aided Design_, vol. CAD-2, no. 3, pp. 202-211, July 1983.

[2]   Wyatt, J.L., Jr., "Signal Delay in RC Mesh Networks," _IEEE Trans. Circuits and Systems_, vol. CAS-32, no. 5, pp. 507-510, May 1985.

[3]   Desoer, Charles A., and Ernest S. Kuh, _Basic Circuit Theory_, McGraw-Hill, 1969.

[4]   Elmore, W.C., "The Transient Response of Damped Linear Networks with Particular Regard to Wideband Amplifiers," _Jour. Appl. Physics_, vol. 19, pp. 55-63, January 1948.

[5]   Lin, T.-M., and C.A. Mead, "Signal Delay in General RC Networks," _IEEE Trans. Computer-Aided Design_, vol. CAD-3, no. 4, pp. 331-349, October 1984.

[6]   Ragunathan, A. and C. Thompson, "Signal Delay in RC Trees with Charge Sharing or Leakage," Report No. UCB/CSD 85/243, Computer Science Division, Univ. of California, Berkeley, CA 94720, June 1985.

[7]   Horowitz, M.A., "Timing Models for MOS circuits," Technical Report, No. SEL83-003, Stanford University, December 1983.

[8]   Tan, H.-N., and J.L. Wyatt, Jr., "Time Optimal Trajectories Associated with Voltage Bounds in RC Tree Networks," M.I.T. VLSI Memo No. 84-205, October 1984.

[9]   Standley, D., and J.L. Wyatt, Jr., (to appear in the M.I.T. VLSI Memo Series).

[10] Wyatt, J.L., Jr., "Monotone Sensitivity of Nonlinear Nonuniform RC Transmission Lines, with Application to Timing Analysis of Digital MOS Integrated Circuits," _IEEE Trans. Circuits and Systems_, vol. CAS-32, no. 1, pp. 28-33, January 1985.

[11] Wyatt, J.L., Jr., "Inequality Theorems for Nonlinear Differential Equations," M.I.T. VLSI Memo No. 82-127, November 1982.

[12] Sandberg, I.W., "A Nonnegativity-Preservation Property Associated with Certain Systems of Nonlinear Differential Equations," _Proc. 1974 IEEE Int. Conf. on Systems, Man, and Cybernetics_, pp. 230-233.

[13] Walter, W., _Differential and Integral Inequalities_, Springer-Verlag, 1970.

[14] Araki, M., "M-Matrices (Matrices with Nonpositive Off-Diagonal Elements and Positive Principal Minors)," Publ. 74/19, Dept. of Computing and Control, Imperial College, London, March 1974.

[15] Markham, T.L., "Nonnegative Matrices Whose Inverses are M-Matrices," Proc. Amer. Math. Soc., vol. 36, no. 2, pp. 326-330, December 1972.

[16] Johnson, C.R., "Inverse M-Matrices," Linear Algebra and Its Applications, vol. 47, pp. 195-216, 1982.

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